

A CHARACTERISTIC EXACT SEQUENCE FOR PRINCIPAL BUNDLES

MARTINA ROVELLI

ABSTRACT. Characteristic classes are invariants for principal bundles that take values in the cohomology of the base space. In the first part of this paper, we propose a uniform interpretation of arbitrary characteristic classes as an obstruction to group reduction and to the existence of an equivariant extension of a universal cocycle. By plugging in the correct parameters, we recover several classical theorems. Afterwards, we construct a long exact sequence of abelian groups for any principal bundle. This sequence involves the cohomology of the base space and the group cohomology of the structure group, and the connecting map is closely related to the characteristic classes of the bundle.

CONTENTS

Introduction	1
1. Preliminaries on principal bundles	5
2. Cohomology & Characteristic classes	11
3. Homotopy fiber of characteristic classes	15
4. Characteristic classes as obstructions	18
5. The category of pointed bundles	24
6. Plus-cohomology groups of principal bundles	29
7. The characteristic long exact sequence	35
References	41

INTRODUCTION

By a classical result of Steenrod [Ste99] and Dold [Dol63], every principal G -bundle E over a space X is classified up to equivalence by the homotopy class of a map $X \rightarrow BG$, where BG is the classifying space of the group. This map is used to produce interesting invariants of bundles. For instance, having fixed an abelian group of coefficients Z , one can look at the *characteristic map* induced in cohomology in degree k :

$$\chi_k : H_{Gp}^k(G; Z) := H^k(BG; Z) \rightarrow H^k(X; Z).$$

Often the group cohomology $H_{Gp}^k(G; Z)$ is known. Its elements are referred to as *universal characteristic classes*, and depend only on the group G . On the other hand, their images in the cohomology of the base space $H^k(X; Z)$ are invariants of the bundle E , known as *characteristic classes* [MS74]. Examples of these are Chern classes, Pontryagin classes, Euler classes and Stiefel-Whitney classes.

For some values of the parameters, it is possible to interpret characteristic classes as obstructions to certain geometric features of the bundle. For example, when $k = 1$, $Z = \mathbb{Z}/2$ and $G = O(n)$, the generator of the group cohomology $H_{Gp}^1(O(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is the *first Stiefel-Whitney class* $w_1 \in H^1(BO(n); \mathbb{Z}/2)$, which measures obstruction to orientability. The property for an $O(n)$ -bundle to be orientable can be expressed by saying that the structure group can be reduced to $SO(n)$ or, equivalently, by saying that the choice of a sign on a distinguished fiber can be extended coherently to the whole bundle. The following theorem is classical [MS74].

Theorem 0.1. For E an $O(n)$ -bundle over a CW-complex X , the following are equivalent.

(A) There exists an $SO(n)$ -bundle \hat{E} over X and an equivalence of $O(n)$ -bundles over X

$$\hat{E} \otimes_{SO(n)} O(n) \cong E.$$

(B) The first Stiefel-Whitney class w_1 of E vanishes,

$$w_1(E) = 0 \in H^1(X; \mathbb{Z}/2).$$

(C) The map $\det : O(n) \rightarrow O(1) \cong \mathbb{Z}/2$ extends to an $O(n)$ -equivariant map $E \rightarrow \mathbb{Z}/2$,

$$\begin{array}{ccc} O(n) & \hookrightarrow & E \\ \det \downarrow & \nearrow & \\ \mathbb{Z}/2 & & \end{array}$$

If the conditions above are verified the bundle is said to be **orientable**.

In the first part of the paper (Sections 3 and 4), we prove an analog of Theorem 0.1 for arbitrary characteristic classes, of which the final statement appears as Corollaries 4.7 and 4.14. For this, we shall identify a topological group and a homomorphism that will play respectively the role of $SO(n)$ and \det in the general case.

Let us first consider the equivalence between (A) and (B), namely the interpretation of the first Stiefel-Whitney class as an obstruction to group reduction. The key fact is that the group $SO(n)$ and the map $B\det : BO(n) \rightarrow B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2; 1)$, which represents the cohomology class $w_1 \in H^1(BO(n); \mathbb{Z}/2)$, fit in a homotopy fibre sequence

$$BSO(n) \longrightarrow BO(n) \xrightarrow{B\det} B\mathbb{Z}/2.$$

This argument is quite standard and applicable to other situations. It is known that analog sequences witness the link between the groups appearing in the Whitehead tower of the orthogonal group $O(n)$ and certain characteristic classes. The consequence is that, for instance, the second Stiefel-Whitney class measures the obstruction to admitting a *spin structure*, the first fractional Pontryagin class measures the obstruction to admitting a *string structure*, and the second fractional Pontryagin class measures the obstruction to admitting a *Fivebrane structure* [SSS09]. In order to abstract the reasoning to the case of an arbitrary universal characteristic class $c \in H^k(BG; \mathbb{Z})$, we construct in Sections 3 and 4 a topological group $\hat{G}(\gamma)$ that fits in a homotopy fiber sequence

$$B\hat{G}(\gamma) \longrightarrow BG \xrightarrow{c} K(\mathbb{Z}; k).$$

Generalizing the classical argument, we prove in Corollary 4.7 that, given a characteristic class $c \in H^k(BG; \mathbb{Z})$ and a G -bundle E over X , the structure group of E can be reduced to $\hat{G}(\gamma)$ if and only if the characteristic class $c(E)$ of E vanishes in $H^k(X; \mathbb{Z})$. Here, by reducing the group of E from G to $\hat{G}(\gamma)$, not only do we mean that the classifying map $X \rightarrow BG$ of E lifts up to homotopy to a map $X \rightarrow B\hat{G}(\gamma)$, but also that E can be written as an honest tensor product

$$E \cong \hat{E} \otimes_{\hat{G}(\gamma)} G$$

of a $\hat{G}(\gamma)$ -bundle \hat{E} with G . Note that the general construction $\hat{G}(\gamma)$ might look different from some of the explicit groups that are known for specific examples. However, there is a homomorphism of topological groups from $\hat{G}(\gamma)$ to the classical model, which is also a weak equivalence. The general approach gives a global explanation of the fact that so many examples of characteristic classes were known to measure obstruction to group reduction with respect to a suitable group. Moreover, for any G and any k , when $c \in H^k(BG; \mathbb{Z})$ is chosen correctly, the group $\hat{G}(\gamma)$ provides a model of the k th layer $G \langle k \rangle$ in the Whitehead tower [Whi52] of G , which is also a group, as stated in Remark 4.4.

On the other hand, for an arbitrary universal characteristic class $c \in H^k(BG; Z)$, the equivalence between (B) and (C) is not as evident in the literature. Even for well studied examples of characteristic classes, there is no obvious choice of homomorphism, for which the characteristic class measures the obstruction to a coherent extension. We define a way to *fatten up* the group G , and more generally any G -bundle, that does not change the homotopy type, and a homomorphism of topological groups

$$\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z.$$

Here, G^{cof} is the fattened-up version of G , and $M^{k-1}Z$ is the $(k-1)$ -fold *Milgram delooping*: an Eilenberg-MacLane space of type $K(Z; k-1)$ that is also a group [Mil67]. Then, we prove in Corollary 4.14 that, given a G -bundle E over X , the cocycle γ extends to the fattened up version E^{cof} of E if and only if the characteristic class $c(E)$ of E vanishes in $H^k(X; Z)$.

In the second part of the paper (Sections 5 and 6), we give a quantitative explanation of the equivalence between (B) and (C) for a characteristic class $c \in H^k(BG; Z)$ represented by a cocycle γ . Indeed, given a G -bundle over X , the vanishing of the characteristic class $c(E)$ can be expressed by saying that c is in the kernel of the characteristic map of E , whereas the extension of the cocycle γ can be expressed by saying that γ is in the image of a certain map. Compactly, this phenomenon is governed by an exact sequence of abelian groups, of which the final statement appears as Corollary 7.3. In order to describe the maps, we need to describe the setup.

We first define a category $\mathcal{Bun}\star$ of bundles, that contains topological groups as bundles over a point, and pointed spaces as bundles with a trivial structure group. In this category lives in particular the *Puppe sequence* of any G -bundle E over X ,

$$\tilde{\Omega}X \longrightarrow G \xrightarrow{\iota} E \xrightarrow{\pi} X \longrightarrow BG,$$

obtained by including the structure group G as a distinguished fiber of E , by projecting E onto the base space X , and by adding the two classifying maps [Rova]. Any topological abelian group, and in particular the Milgram delooping M^kZ , is a monoid in this category, and thus the homset $\mathcal{Bun}\star(E, M^kZ)$ becomes an abelian group, when endowed with pointwise multiplication. We define on it a congruence \simeq_+ , and focus on the *plus-cohomology group* of E :

$$H_+^k(E; Z) := \mathcal{Bun}\star(E^{\text{cof}}, M^kZ) / \simeq_+.$$

When the input is a discrete bundle X , namely a pointed space, or a codiscrete bundle G , namely a topological group, the plus cohomology groups recover the ordinary (group) cohomology groups, i.e.,

$$H_+^k(X; Z) \cong H^k(X; Z) \text{ and } H_+^k(G; Z) \cong H^{k+1}(BG; Z).$$

More generally, the-plus cohomology of a trivial bundle $X \times G$ gives the sum of the two, and the plus-cohomology of a contractible bundle EG is trivial, i.e.,

$$H_+^k(X \times G; Z) \cong H^k(X; Z) \oplus H^{k+1}(BG; Z) \text{ and } H_+^k(EG; Z) \cong \{0\}.$$

As the equivalence relation \simeq_+ is not totally well behaved with respect to precomposition of maps, the assignment $E \mapsto H_+^k(E; Z)$ will not define a functor on the category $\mathcal{Bun}\star$. However, it acts on the Puppe sequence of a bundle E , and induces in fact a long exact sequence of abelian groups

$$\dots \longrightarrow H_+^{k-1}(E; Z) \longrightarrow H_{\mathcal{G}_p}^k(G; Z) \xrightarrow{\chi_k} H^k(X; Z) \longrightarrow H_+^k(E; Z) \longrightarrow \dots,$$

of which the connecting map is the characteristic map χ_k . This is the main contribution of the paper. Note that the process is dual to taking the homotopy groups of the Puppe sequence, which produces the well known homotopy long exact sequence of the bundle.

The exactness in $H_{\mathcal{G}_p}^k(G; Z)$ reflects the fact that characteristic classes are obstructions to the extension of a cocycle to the whole bundle, as mentioned before. The exactness in $H_+^k(E; Z)$ says

that the plus cohomology group is a non-trivial extension of the kernel of the characteristic map in degree $k + 1$, and the cokernel of the characteristic map in degree k ,

$$\{0\} \longrightarrow \text{coker}(\chi_k) \longrightarrow H_+^k(E; Z) \longrightarrow \ker(\chi_{k+1}) \longrightarrow \{0\},$$

which carry, respectively, information about the characteristic classes of the bundles in degree k and $k + 1$. This is an indication of some surprising interaction between characteristic classes in different degrees. Moreover, the fiber $H_+^{k-1}(E; Z)_c$ over a universal characteristic class $c \in H_{\mathcal{G}_p}^k(G; Z)$,

$$H_+^{k-1}(E; Z)_c := \text{fib}_c(H_+^{k-1}(E; Z) \rightarrow H_{\mathcal{G}_p}^k(G; Z)),$$

is related to the *twisted cohomology* $H_{[E]}^{k-1}(X; Z)$ of X with respect to the bundle E [NSS15], which counts the reductions of E to a $\hat{G}(\gamma)$ -bundle. More precisely, there is an action [Rovb] of the group cohomology $H_{\mathcal{G}_p}^{k-1}(G; Z)$ on the twisted cohomology $H_{[E]}^{k-1}(X; Z)$, such that the quotient is exactly the fiber,

$$H_+^{k-1}(E; Z)_c \cong H_{[E]}^{k-1}(X; Z) / H_{\mathcal{G}_p}^{k-1}(G; Z).$$

The action, together with the isomorphism, provide useful information on the cardinality of the twisted cohomology, namely on the number of structures of a specific kind on a bundle.

Digression 0.2. It was pointed out by Jeffrey Carlson (personal communication, November 15, 2016) that there is an alternative approach to produce a long exact sequence involving the characteristic map. Indeed, the cofiber sequence

$$X \xrightarrow{t} BG \longrightarrow C(t) \longrightarrow \Sigma X$$

induces a long exact sequence

$$\dots \longrightarrow H^k(C(t); Z) \longrightarrow H_{\mathcal{G}_p}^k(G; Z) \xrightarrow{\chi_k} H^k(X; A) \longrightarrow H_+^{k+1}(C(t); Z) \longrightarrow \dots$$

There is a comparison map $H_+^{k+1}(C(t); Z) \rightarrow H_+^k(E; Z)$ [Rovb]. Using the fact that the plus-cohomology group $H_+^k(E; Z)$ and the ordinary cohomology group $H^{k+1}(C(t); Z)$ of the cofiber $C(t)$ both fit into long exact sequences, the comparison map can then be proven to be an isomorphism,

$$H_+^k(E; Z) \cong H^{k+1}(C(t); Z).$$

This isomorphism gives an interpretation of the plus-cohomology groups in purely homotopical terms.

In Section 1 we recall the background on principal bundles, and in Section 2 we introduce characteristic classes and explain some properties. In particular we provide an unusual description of the ordinary cohomology groups that will be useful later in the paper. In Section 3 we construct the homotopy fiber of a characteristic class, and prove that it is a countable CW-complex, which will be crucial in order to define $\hat{G}(\gamma)$. In Section 4 we prove that characteristic classes measure the obstruction to group reduction with respect to $\hat{G}(\gamma)$, and the obstruction to the extension of the cocycle γ . In Section 5 we define the category $\mathcal{B}un\star$ of bundles, and study the classes of maps of type k and w , which will be used to define the equivalence relation \simeq_+ , and in Section 6 we define the plus-cohomology groups. Finally, in Section 7 we construct the long exact sequence associated to a principal bundle.

Acknowledgements. I would like to thank Kathryn Hess for her support and for her careful reading of this paper. I am grateful to Jeffrey Carlson for suggesting an alternative description of the plus-cohomology groups, and for providing insightful ideas on how to build the comparison.

1. PRELIMINARIES ON PRINCIPAL BUNDLES

In this section we recall the basics about principal bundles and their classification(s) up to equivalence. Note that the notions and results stated in this sections are the pointed versions of already existing notions and results.

We start with the classical (unpointed) notions of *principal bundle* and *equivalence of principal bundles* [Ste99].

Terminology 1.1. Let G be a topological group, X a (pointed) topological space, and E a topological space together with a map $E \rightarrow X$ and a right G -action $E \times G \rightarrow E$.

- An open subset $U \subset X$ is a **trivialization open subset** of X for E if there is a homeomorphism

$$E|_U \cong U \times G$$

that is compatible with the right G -action and with the map to X .

- An open cover $\{U\}$ of X is a **trivialization open cover** if it contains only trivialization open subsets of X for E .

Definition 1.2. Let G be a topological group, and X a (pointed) topological space. A G -**bundle** over X is a topological space E together with a map $E \rightarrow X$ and a right G -action $E \times G \rightarrow E$ such that there exist a trivialization open cover of X for E . We denote by ${}_X\mathcal{B}un_G$ the class of G -bundles over X .

Definition 1.3. Let E and E' be G -bundles over X . An **equivalence** between E and E' is a homeomorphism $\phi : E \rightarrow E'$ that is right G -equivariant and respects the projection over X . The bundles E and E' are said to be **equivalent** if there exists an equivalence between them, in which case we write $E \simeq E'$.

If E is a G -bundle over X , every fiber is isomorphic to G as a right G -space. However, there is no canonical identification of any fiber with the structure group. In particular, there is no canonical way to embed the group G into E . As we will want a distinguished inclusion of $G \hookrightarrow E$, dual to the projection $E \rightarrow X$, we focus on bundles that are more structured.

If X is pointed in x_0 , choosing an equivariant identification of the fiber $E|_{x_0} := \pi^{-1}(x_0)$ with the group G is equivalent to identifying in the distinguished fiber $E|_{x_0}$ a point e_0 with the identity of the group e_G . With the choice of the base point e_0 , the bundle E becomes a *pointed bundle*.

Definition 1.4. Let G be a topological group, and X a pointed topological space. A **pointed G -bundle over X** is a pointed topological space E together with a pointed map $E \rightarrow X$ and a right G -action $E \times G \rightarrow E$ such that there exist a trivialization open cover of X for E . We denote by ${}_X\mathcal{B}un_\star G$ the class of G -bundles over X .

Definition 1.5. Let E and E' be pointed G -bundles over X . A **pointed equivalence** between E and E' is a pointed homeomorphism $\phi : E \rightarrow E'$ that is right G -equivariant and respects the projection over X . The bundles E and E' are said to be **pointed equivalent** if there exists a pointed equivalence between them, in which case we write $E \cong E'$.

Remark 1.6. Note that every pointed equivalence of principal bundles is bijective, and the inverse function is itself a pointed equivalence. It follows that pointed equivalence of pointed bundles is an equivalence relation.

Many results and constructions extend naturally to the pointed framework, by endowing the spaces with the correct base point.

Convention 1.7. Unless otherwise specified, we apply the following conventions.

- » The base point of a pointed space X is denoted x_0 .
- » A group G is considered to be pointed in the identity e_G .
- » The standard interval $I := [0, 1]$ is considered to be pointed in 0.

- » The product $X \times Y$ is considered to be pointed in (x_0, y_0) .
- » A subspace $Z \subset X$ containing x_0 is considered to be pointed in x_0 .
- » A quotient X/\sim of X is considered to be pointed in $[x_0]$.
- » A loop space ΩX is considered to be pointed in the constant path to x_0 .

As anticipated, pointed bundles have a distinguished inclusion of the structure group.

Remark 1.8. If E is a pointed G -bundle over X , the assignment $e_G \mapsto e_0$ uniquely determines a right G equivariant map $G \rightarrow E$, given by $g \mapsto e_0 \cdot g$. In particular we can build the diagram

$$G \xrightarrow{\iota} E \xrightarrow{\pi} X$$

It plays the role of a *short exact sequence*, and reflects a certain selfduality of the data.

As every bundle over a CW-complex is a Hurewicz fibration [Spa81], the following result will be useful.

Theorem 1.9. [Spa81, Theorem 10, Section 7.2] Let X be a CW-complex, E a pointed space and $p : E \rightarrow X$ a Hurewicz fibration. Then, if $F := p^{-1}(x_0)$ is the fiber, the maps

$$F \xrightarrow{\iota} E \xrightarrow{p} X$$

induce a natural long exact sequence of pointed sets,

$$\dots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \dots,$$

which are in fact abelian groups in positive degree. \square

It is possible to change the base space of a bundle or the structure group by *pulling back* or *pushing forward*. These constructions have been widely used in the literature on principal bundles. For instance, they both appear in [Ste99], in Chapters 10 and 3 respectively. They are also known by other names, e.g., *induced bundle* for the pullback construction, and *balanced product*, *associated bundle*, or just *tensor product* for the pushforward construction.

Proposition 1.10. (a) The **pullback construction**, defined by

$$(f, E) \mapsto f^* E := X' \times_X E,$$

gives a map

$$\mathcal{Top}_*(X', X) \times_X \mathcal{Bun}\star_G \longrightarrow_{X'} \mathcal{Bun}\star_G.$$

It is well defined up to pointed equivalence, i.e.,

$$\mathcal{Top}_*(X', X) \times_X \mathcal{Bun}\star_G / \cong \longrightarrow_{X'} \mathcal{Bun}\star_G / \cong.$$

(b) The **pushforward construction**, defined by

$$(a, E) \mapsto a_* E := E \otimes_G G',$$

gives a map

$$\mathcal{Gp}(\mathcal{Top})(G, G') \times_X \mathcal{Bun}\star_G \longrightarrow_X \mathcal{Bun}\star_{G'},$$

It is well defined up to pointed equivalence, i.e.,

$$\mathcal{Gp}(\mathcal{Top})(G, G') \times_X \mathcal{Bun}\star_G / \cong \longrightarrow_X \mathcal{Bun}\star_{G'} / \cong.$$

\square

The following properties are easy to verify.

Proposition 1.11. Let E be a pointed G -bundle over X .

(a) For every $g : X'' \rightarrow X'$ and $f : X' \rightarrow X$, there is a pointed equivalence of pointed bundles

$$(f \circ g)^* E \cong g^* f^* E.$$

- (b) For every $a : G \rightarrow G'$ and $b : G' \rightarrow G''$, there is a pointed equivalence of pointed bundles

$$(b \circ a)_* E \cong b_* a_* E.$$

□

- (c) For every $a : G \rightarrow G'$ and $f : X' \rightarrow X$, there is a pointed equivalence of pointed bundles

$$f^* a_* E \cong a_* f^* E.$$

Proposition 1.12. Let E be a pointed G -bundle over X .

- (a) For every $g : X' \rightarrow X$ there is a pointed G -equivariant map

$$\epsilon_g : g^* E \rightarrow E,$$

that induces g on the base spaces.

- (b) For every $a : G \rightarrow G'$ there is a pointed map over X

$$\eta_a : E \rightarrow a_* E,$$

that is a -equivariant.

□

Proposition 1.13. Let E be a pointed G -bundle over X and E' a pointed G' -bundle over X' . Let

$$\psi : E \rightarrow E'$$

be a pointed map that is equivariant with respect to a map $a : G \rightarrow G'$ of topological groups and induces a pointed map $g : X \rightarrow X'$.

- (1) There is a pointed equivalence of pointed G' -bundles over X

$$a_* E \xrightarrow{\cong} g^* E'.$$

- (2) If X is (path-)connected, then if two of ψ , a , and g are weak equivalences, so is the third.

□

Classically, up to equivalence every bundle is obtained by pulling back the *universal bundle* of the structure group, and by pushing forward the *universal bundle* of the base space. The models that we will use, of which the construction is due to Milnor, can be defined only for spaces or groups that satisfy certain conditions, which we now introduce.

Definition 1.14. (a) A pointed space is **nice** if it is a countable CW-complex and the base point is a vertex.
 (b) A topological group is **nice** if it is a countable CW-group, namely if it is a countable CW-complex and the group structure maps send the k -th skeleton to the k -skeleton for every $k \geq 0$.

As we will be working with pointed spaces, the following fact will be useful.

Proposition 1.15. Let X and X' be CW-complexes pointed on a vertex. Every pointed weak equivalence

$$f : X \xrightarrow{\sim} X'$$

is a based homotopy equivalence.

Lemma 1.16. Let \mathcal{M} be a model category, $f, f' : X \rightarrow Y$ morphisms in \mathcal{M} , and CX a good cylinder for X . If there exists a very good left homotopy between f and f' , then there exists a left homotopy between f and f' with respect to the cylinder CX . □

Proof of Proposition 1.15. By [Hir, Theorem 2.7], there is a model structure for \mathcal{Top}_* where fibrations, cofibrations and weak equivalences are created in \mathcal{Top} with the Serre model structure by forgetting the base point. With respect to this model structure, the space $\frac{I \times X}{I \times \{x_0\}}$ is a good cylinder for X . Moreover, with respect to this model structure, CW-complexes pointed on a vertex are fibrant and cofibrant, and pointed weak equivalences are weak equivalences. So the weak

equivalence $f : X \rightarrow X'$ is a homotopy equivalence in \mathcal{Top}_* . Thus, by the Whitehead Theorem, f is a left homotopy equivalence in \mathcal{Top}_* . This means that there exists $g : X' \rightarrow X$ such that

$$(*) \quad f \circ g \sim_l \text{id} \text{ and } g \circ f \sim_l \text{id},$$

namely, $f \circ g$ and $g \circ f$ are left homotopic to identity maps. By [DS95, Lemma 4.6], there are very good left homotopies that witness (*), and by Lemma 1.16, there are homotopies with respect to the good cylinder $\frac{I \times X}{I \times \{x_0\}}$, namely based homotopies. It follows that f and g are based homotopy inverse to each other, and f is a based homotopy equivalence. \square

Let X be a connected nice pointed space. By [Whi49, Theorem 13], there exists a locally finite, countable simplicial complex X' and an (unbased) homotopy equivalence $s : X \rightarrow X'$. Up to further subdivision of X' , which does not change the topology because X' is locally finite, we can assume that the image of the base point through s is a vertex of X' . Then s is a pointed weak equivalence between CW-complexes, and therefore a based homotopy equivalence by Proposition 1.15. In [Mil56a], for any locally finite, countable simplicial complex X' Milnor constructs a nice topological group $\tilde{\Omega}X'$, and a contractible $\tilde{\Omega}X'$ -bundle $\tilde{P}X'$ over X' . We then let $\tilde{\Omega}X := \tilde{\Omega}X'$ and $\tilde{P}X := \tilde{P}X'$. As the constructions involve the choice of a simplicial approximation s , they are not functorial on (nice) pointed spaces and pointed continuous maps. However, we recall in Remark 1.20 that it is possible to define an action of $\tilde{\Omega}$ on pointed maps between nice pointed spaces that is *pseudofunctorial* with respect to the notion of *based algebraic equivalence* (Definition 1.18). The key property of $\tilde{P}X$ is recalled in Theorem 1.19.

On the other hand, Milnor considers in [Mil56b] the join of infinitely many copies of G , which is the underlying set of the fat realization of the topological Bar construction $\text{Bar}_\bullet(G; G)$, of which the n -th degree is given by G^n . He endows it with two different topologies: the *weak topology*, which coincides with the topology of the fat realization

$$E^w G := ||\text{Bar}_\bullet(G; G)||,$$

and the *strong topology*, which defines a different topological space $E^s G$. The space $E^s G$ has a natural (right) action of G and is in fact a contractible G -bundle over the quotient $B^s G := E^s G / G$ [Mil56b, Section 3]. Both $E^s G$ and $B^s G$ are functorial.

In this paper we will mostly use the second space $E^w G$, which has good properties when G is nice. The fact that G admits a countable decomposition ensures that the action of G is still continuous, as pointed out in the proof of [Mil56b, Theorem 5.1], and $E^w G$ becomes a contractible G -bundle [Mil56b, Theorem 5.1] over the quotient $B^w G = E^w G / G$, which is proven in [ML70] to be the fat realization

$$B^w G = ||\text{Bar}(\star; G)||.$$

Moreover, $E^w G$ and $B^w G$ can be given countable CW-structures, too. Both $E^w G$ and $B^w G$ are functorial on nice topological groups.

Remark 1.17. If G is a nice topological group, the identity function yields a continuous map $\text{id} : E^w G \rightarrow E^s G$, because the strong topology is contained in the weak topology. As it is in fact a map of G -bundles between two contractible spaces, by Theorem 1.9, the map induced on the base spaces $\text{id} : B^w G \rightarrow B^s G$ is a pointed weak equivalence. Note that this map is also natural. As $B^s G$ does not have (a priori) the homotopy type of a CW complex, $B^s G$ and $B^w G$ might not be homotopy equivalent.

The following two-fold classification of pointed bundles up to pointed equivalence is the analog of what proven in [Rov] in the unpointed context. The details of the pointed case are spelled out in [Rovb].

Definition 1.18. Let G and G' be topological groups. Two morphisms $a, b : G \rightarrow G'$ are said to be **based algebraically equivalent** if $B^w a$ is based homotopic to $B^w b$, in which case we write $a \equiv_\star b$.

Theorem 1.19. Let X be a nice pointed connected space and G a nice topological group. Then the assignments

$$[a : \tilde{\Omega}X \rightarrow G] \mapsto a_* \tilde{P}X \text{ and } [g : X \rightarrow B^w G] \mapsto f^* E^w G$$

induce bijections

$$\mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G)/\equiv_* \cong {}_X \mathcal{B}un_G/\cong \cong \mathcal{T}op_*(X, B^w G)/\simeq_*.$$

□

Note that the restriction for X to be a *countable* CW-complex is needed only for the bijection on the left-hand side bijection. The other one holds even if X is not necessarily countable, as recalled in Theorem 1.26.

Remark 1.20. If $f : X \rightarrow Y$ is a pointed map between nice pointed spaces, by Theorem 1.19 there exists a morphism

$$\tilde{\Omega}X \rightarrow \tilde{\Omega}Y$$

that classifies the $\tilde{\Omega}Y$ -bundle $f^* \tilde{P}Y$ over X ,

$$(\tilde{\Omega}f)_* \tilde{P}X \cong f^* \tilde{P}Y.$$

Note that the homomorphism $\tilde{\Omega}f$ is unique up to based algebraic equivalence. This action of $\tilde{\Omega}$ on pointed maps of nice pointed spaces is pseudofunctorial with respect to based algebraic equivalence.

With respect to the actions of B^w and $\tilde{\Omega}$ on arrows described above, the bijections of Theorem 1.19 are natural.

Since Theorem 1.19 expresses an up-to-homotopy adjunction, there are notions of *unit* and *counit*.

Definition 1.21. (a) If X is a connected nice pointed space, the **unit** of X is a pointed map

$$\eta_X : X \rightarrow B^w \tilde{\Omega}X$$

such that

$$\eta_X^* E^w \tilde{\Omega}X \cong \tilde{P}X.$$

(b) If G is a nice topological group, the **counit** of G is a map of topological groups

$$\epsilon_G : \tilde{\Omega}B^w G \rightarrow G$$

such that

$$\epsilon_{G*} \tilde{P}B^w G \cong E^s G.$$

Remark 1.22. Note that the unit of a connected nice pointed topological space X is the map induced on the base spaces by the map of $\tilde{\Omega}X$ -bundles $\tilde{P}X \cong \eta_X^* E^w \tilde{\Omega}X \rightarrow E^w \tilde{\Omega}X$. By Theorem 1.9, the unit η_X is a pointed weak equivalence, and therefore a based homotopy equivalence by Proposition 1.15.

Similarly, the counit of a nice topological group G is the map induced on the distinguished fibers by the map of principal bundles over $B^w G$ given by $\tilde{P}B^w G \rightarrow \epsilon_{G*} \tilde{P}B^w G \cong E^w G$. By Theorem 1.9, the counit ϵ_G is a pointed weak equivalence, and therefore a based homotopy equivalence by Proposition 1.15.

While countable discrete groups are nice, many important examples of topological groups are (a priori) not. In particular, it is not known whether matrix groups, e.g. the orthogonal group $O(n)$, can be given the structure of a nice topological group. Since we do not want to exclude these groups of matrices, we construct ad hoc universal bundles for them, given by *Grassmannians*.

Definition 1.23. A topological group is a **matrix group** if it is one of the following:

- » the **orthogonal group** $O(n)$, for $n \geq 0$.
- » the **special orthogonal group** $SO(n)$, for $n \geq 0$.
- » the **unitary group** $U(n)$, for $n \geq 0$.
- » the **special unitary group** $SU(n)$, for $n \geq 0$.

For G a matrix group, the **Stiefel manifold** $St(G)$ is the space of n -frames in \mathbb{R}^∞ or \mathbb{C}^∞ that are compatible with G [Ste99],[MS74]. When $G = O(n), U(n), SO(n)$ we obtain respectively the **real Stiefel manifold**, the **complex Stiefel manifold**, and the **real oriented Stiefel manifold**. It has a natural (right) action of G , of which the quotient is the **Grassmann manifold** $Gr(G)$. When $G = O(n), U(n), SO(n)$ we obtain respectively the **real Grassmann manifold**, i.e., the space of n -dimensional subspaces of \mathbb{R}^∞ , the **complex Grassmann manifold**, i.e., the space of n -dimensional subspaces of \mathbb{C}^∞ , and the **real oriented Grassmann manifold**, i.e., the space of n -dimensional oriented subspaces of \mathbb{R}^∞ . Moreover $St(G)$ is a contractible G -bundle over $Gr(G)$ [MS74, Chapter 5]. We assume $Gr(G)$ and $St(G)$ to be pointed in the 0-dimensional subspace.

Proposition 1.24. For a matrix group G , the Grassmannian $Gr(G)$ is a nice pointed space.

Proof. A cell decomposition for $Gr(G)$ is constructed in [MS74, Chapter 6], so $Gr(G)$ is a CW-complex. By reading through the definition of the cells, we see that the cells can be enumerated by means of the *Schubert symbols*, which are certain finite collections of natural numbers, and therefore there are countably many. Thus $Gr(G)$ is a countable CW-complex. The base point is a vertex. \square

In particular, the Stiefel manifold is a *universal bundle*, according to the following definition.

Definition 1.25. Let G be a topological group. If a pointed G -bundle \tilde{E} over a nice pointed space \tilde{B} is contractible, we say that \tilde{E} is a **universal bundle** and \tilde{B} is a **classifying space** for G . If then E is a pointed G -bundle over a CW-complex, there exists a classifying map $t : X \rightarrow \tilde{B}$ such that $t^*\tilde{E} \cong E$.

By the following classical theorem, all classifying spaces are homotopy equivalent.

Theorem 1.26 ([Dol63]). Let X be a connected CW-complex, and G a topological group, \tilde{B} a CW-complex and \tilde{E} a universal pointed G -bundle. Then the assignment

$$[g : X \rightarrow \tilde{B}] \mapsto f^*\tilde{E}$$

induce a bijection

$$\mathcal{Top}_*(X, Gr(G))/\simeq_* \cong {}_X\mathcal{Bun}_G/\cong.$$

\square

Thanks to the following theorem, we can make sense of the counit of a matrix group.

Theorem 1.27 ([Mil56a, Theorem 5.1]). Let X be a connected nice pointed space and G a topological group. Then the assignment

$$[a : \tilde{\Omega}X \rightarrow G] \mapsto a_*\tilde{P}X$$

induces a surjection

$$\mathcal{Gp}(\mathcal{Top})(\tilde{\Omega}X, G) \twoheadrightarrow {}_X\mathcal{Bun}_G/\cong.$$

\square

Definition 1.28. If G is a matrix group, the **counit** of G is a map of topological groups

$$\epsilon_G : \tilde{\Omega}Gr(G) \rightarrow G$$

such that

$$\epsilon_{G*}\tilde{P}Gr(G) \cong St(G).$$

Notation 1.29. Let G be a nice topological group or a matrix group.

- The **classifying space** is the nice pointed space

$$BG := \begin{cases} B^wG & \text{if } G \text{ is a nice topological group, and} \\ Gr(G) & \text{if } G \text{ is a matrix group.} \end{cases}$$

- The **classifying bundle** is the G -bundle over BG

$$EG := \begin{cases} E^w G & \text{if } G \text{ is a nice topological group, and} \\ St(G) & \text{if } G \text{ is a matrix group.} \end{cases}$$

As a consequence of the homotopy long exact sequence induced by a fibration Theorem 1.9, we have can describe the homotopy groups of Milnor's loop groups, Milnor's classifying spaces and Grassmannians.

Proposition 1.30. (a) For a nice pointed topological space X ,

$$\pi_{i-1}(\tilde{\Omega}X) \cong \pi_i(X).$$

(b) For a nice topological group or a matrix group G ,

$$\pi_{i+1}(BG) \cong \pi_i(G).$$

□

2. COHOMOLOGY & CHARACTERISTIC CLASSES

The aim of this section is to recall and elaborate on *characteristic classes*, which are important invariants for principal bundles.

Proposition 2.1. Let A be a topological abelian group. Then the assignment

$$X \mapsto [X, A]_\star := \mathcal{Top}_\star(X, A)/\simeq_\star$$

defines a functor

$$[-, A]_\star : \mathcal{Top}_\star^{\text{op}} \rightarrow \mathcal{Ab},$$

where the multiplication in $[X, A]_\star$ is defined pointwise. □

We will be interested in the case when $A := M^k Z$ is the k -fold *Milgram delooping* of an abelian group Z . In [Mil67], Milgram associates to any H-space A a topological space MA , that also agrees with Stasheff's original model [Sta78]. If A is in fact an (abelian) group, then it is mentioned in [BH98] that MA is the ordinary realization of the topological Bar construction of A ,

$$MA = |\text{Bar}(\star; A)|.$$

Proposition 2.2. Let A be an abelian nice topological group.

- (1) MA is an abelian nice topological group.
- (2) There is a homotopy natural based homotopy equivalence

$$B^w A \rightarrow MA.$$

As a consequence, $\pi_{i+1}(MA) \cong \pi_i(A)$.

- (3) There is a pointed map of H-spaces $A \rightarrow \Omega MA$ that is also a based homotopy equivalence.
- (4) M respects products, and in particular

$$M(A \times A) \cong MA \times MA.$$

Proof. (1) The fact that MA is a topological abelian group is proven in [Mil67, Corollary 1.7].

In [Mil67, Theorem 2.3], a cell decomposition for MA is constructed. As A is a countable CW-complex, the cell decomposition constructed in [Mil67, Theorem 2.3] recovers the topology, so MA is a CW-complex. By reading through the definition of the cells, we see that the cells can be enumerated by finite collections of cells of A , and therefore there are countably many. Therefore MA is a countable CW-complex. Moreover, it is proven in [Mil67, Section 2] that the multiplication on MA defined in [Mil67, Corollary 1.7] is continuous and cellular. The fact that the inversion is cellular is obvious, based on the explicit formula for inverses, in the proof of [Mil67, Corollary 1.7]. So MA is an abelian nice topological group.

- (2) As the unit of A is a vertex, the simplicial space $\text{Bar}(\star; A)$ is good (according to [Seg74, Definition A.4]). Then, by [Seg74, Proposition A.1(iv)], the Bousfield-Kan map

$$B^w A = ||\text{Bar}(\star; A)|| \rightarrow |\text{Bar}(\star; A)| = MA$$

is a pointed weak equivalence, and therefore a based homotopy equivalence by Proposition 1.15.

- (3) The map is constructed and proven to be a weak equivalence in [Mil67, Section 5]. It is pointed. By [FP90, Corollary 5.3.7] ΩMA has the (unbased) homotopy type of a CW-complex, so there is a CW-complex and an (unbased) homotopy equivalence $\Omega MA \rightarrow C$. By [FP90, Corollary 2.3.7], the map can be assumed to be pointed. By Proposition 1.15, the map is in fact a based homotopy equivalence. Moreover, also the composite $A \rightarrow \Omega MA \rightarrow C$ is a pointed weak equivalence, and therefore a based homotopy equivalence by Proposition 1.15. By the two-out-of-three property, the map $A \rightarrow \Omega MA$ is a based homotopy equivalence.
- (4) As A is nice, the Bar construction $\text{Bar}_\bullet(\star, A)$ is a simplicial space that is degreewise a CW-complex and has cellular faces and degeneracies. By [May72, Corollary 11.6], the realization respects products of such simplicial spaces. Alternatively, this is also mentioned in [Seg74, §3].

□

By specializing Propositions 2.1 and 2.2 to $A = M^k Z$, the **k -fold Milgram delooping**

$$M^k Z := \underbrace{M \circ \cdots \circ M}_{k \text{ times}}(Z) \in \mathcal{Ab}(\mathcal{Top})$$

of an abelian group Z , we obtain the following corollaries.

Corollary 2.3. Let Z be an abelian countable group, and $k \geq 0$.

- (1) $M^k Z$ is an abelian nice topological group.
- (2) There is a homotopy natural based homotopy equivalence

$$B^w M^k Z \rightarrow M^{k+1} Z.$$

As a consequence, $M^k Z \simeq K(Z; k)$ is an Eilenberg-MacLane space.

- (3) There is a pointed map of H-spaces $M^k Z \rightarrow \Omega M^{k+1} Z$ that is also a based homotopy equivalence. □

Corollary 2.4. Let Z be an abelian group and $k \geq 0$. Then the assignment

$$X \mapsto [X, M^k Z]_\star,$$

defines a functor

$$[-, M^k Z]_\star : \mathcal{Top}_\star^{\text{op}} \rightarrow \mathcal{Ab},$$

where the multiplication in $[X, M^k Z]_\star$ is defined pointwise. □

The ordinary cohomology groups with coefficients in a group Z are usually defined in a similar fashion, where the role of the topological group $M^k Z$ is played by the loop space $\Omega K(Z; k+1)$ of an Eilenberg-MacLane space. The key fact is that, for every pointed topological space Y , the loop space ΩY , together with the concatenation of paths, is an *H-group*.

Definition 2.5. [Pic92, Section 1.2] A pointed topological space H is a **H-group** if it is endowed with a **pointed multiplication** $\mu_H : H \times H \rightarrow H$, and a **pointed inversion** $j : H \rightarrow H$, such that

- $\mu \circ (H \times \mu) \simeq_\star \mu \circ (\mu \times H)$,
- $\mu \circ (H, h_0) \simeq_\star \text{id}_H$ and $\mu \circ (h_0, H) \simeq_\star \text{id}_H$,
- $\mu \circ (H, j) \simeq_\star h_0$ and $\mu \circ (j, H) \simeq_\star h_0$.

Proposition 2.6 and Theorem 2.7 are the analog of Proposition 2.1 and Corollary 2.4 respectively.

Proposition 2.6. [Pic92, Theorem 1.2.5] Let Y be a pointed topological space. Then the assignment

$$X \mapsto [X, \Omega Y]_* := \mathcal{Top}_*(X, \Omega Y) / \simeq_*$$

defines a functor

$$[-, \Omega Y]_* : \mathcal{Top}_*^{\text{op}} \rightarrow \mathcal{Gp},$$

where the multiplication in $[X, \Omega Y]_*$ is defined pointwise. \square

Theorem 2.7. [Hat02, Theorem 4.57] Let Z be an abelian group and $k \geq 0$. Then the assignment

$$X \mapsto H^k(X; Z) =: [X, \Omega K(Z; k+1)]_*,$$

defines an ordinary cohomology theory with coefficients in Z

$$H^k(-; Z) : \mathcal{Top}_*^{\text{op}} \rightarrow \mathcal{Ab}.$$

\square

We now prove that the group $[X, M^k Z]_*$ of Corollary 2.4 coincides with the ordinary cohomology group $H^k(X; Z)$.

Theorem 2.8. Let Z be an abelian group, and $k > 0$. For every topological space X there is a natural isomorphism of abelian groups

$$H^k(X; Z) \cong [X, \Omega K(Z; k+1)]_* \cong [X, M^k Z]_* \in \mathcal{Ab}$$

between ordinary cohomology group and the plus-cohomology group.

The proof requires some preliminary results.

Definition 2.9. Let H and H' be H-groups (assumed to be pointed in the identity). A **morphism of H-groups** is a pointed map $a : H \rightarrow H'$ such that

$$a \circ \mu_H \simeq_* \mu_{H'} \circ (a \times a).$$

Lemma 2.10. Let X be a pointed topological space, and H and H' be H-groups. Given a morphism of H-groups $a : H \xrightarrow{\sim} H'$ that is also a based homotopy equivalence, the induced map $a \circ -$ is an isomorphism of groups

$$a \circ - : [X, H]_* \xrightarrow{\cong} [X, H']_* \in \mathcal{Gp}.$$

Proof of Lemma 2.10. We prove that $a \circ -$ is a group homomorphism. For every pointed maps $f, f' : X \rightarrow H$ we have that

$$\begin{aligned} a \circ (f \cdot f') &:= a \circ \mu_H \circ (f, f') \\ &\simeq_* \mu_{H'} \circ (a \times a) \circ (f, f') \\ &= \mu_{H'} \circ (a \circ f, a \circ f') \\ &= (a \circ f) \cdot (a \circ f'). \end{aligned}$$

We prove that $a \circ -$ is bijective. Take b a based homotopy inverse for a . Then $b \circ -$ represents the inverse function for $a \circ -$. \square

Lemma 2.11. Let X and X' be pointed topological spaces. Given a based homotopy equivalence of based spaces $f : X \xrightarrow{\sim} X'$, the induced map Ωf is a morphism of H-groups that is also a based homotopy equivalence

$$\Omega f : \Omega X \xrightarrow{\sim} \Omega X'.$$

\square

Lemma 2.12. Let Z be an abelian group and $n > 0$. If Y and Y' are CW-complexes pointed in a vertex and Eilenberg-MacLane spaces of type $K(n; Z)$, then they have the same based homotopy type,

$$Y \simeq_* Y'$$

Proof of Lemma 2.12. By [Hat02, Prop 4.30] we know that Y and Y' have the same (unbased) homotopy type. Namely, there exist maps $f : Y \rightarrow Y'$ and $g : Y' \rightarrow Y$ that are homotopy inverse to each other,

$$f \circ g \simeq \text{id}_{Y'} \text{ and } g \circ f \simeq \text{id}_Y.$$

Then, it is mentioned in [Hat02, Section 4.3] that, for every connected pointed topological space X and pointed Eilenberg-MacLane space $K(Z, n)$ the following bijection holds

$$[X, K(Z, n)] \cong [X, K(Z, n)]_\star.$$

In particular, every map $X \rightarrow K(Z, n)$ is homotopic to a pointed map and if two pointed maps $X \rightarrow K(Z, n)$ are homotopic then they are in fact based homotopic. By taking X and $K(Z, n)$ to be Y and Y' in the two possible ways we obtain that f and g are homotopic to pointed maps \tilde{f} and \tilde{g} respectively,

$$f \simeq \tilde{f} \text{ and } g \simeq \tilde{g},$$

and \tilde{f} and \tilde{g} are based homotopy inverse to each other,

$$\tilde{f} \circ \tilde{g} \simeq_\star \text{id}_{Y'} \text{ and } \tilde{g} \circ \tilde{f} \simeq_\star \text{id}_Y.$$

□

Proof of Theorem 2.8. We have the following isomorphisms of groups:

$$\begin{aligned} H^k(X; Z) &\cong [X, \Omega K(k+1; Z)]_\star && \text{Theorem 2.7} \\ &\cong [X, \Omega M^{k+1} Z]_\star && \text{Corollary 2.3(3), and Lemmas 2.10, 2.12, 2.11} \\ &\cong [X, M^k Z]_\star. && \text{Corollary 2.3(3)} \end{aligned}$$

□

We now recall standard terminology concerning characteristic classes.

Terminology 2.13. Let X be a CW-complex, G a nice topological group or a matrix group, Z an abelian group, $k \geq 0$, and E a pointed G -bundle over X .

- The **k th group cohomology group** of G is the group

$$H_{Gp}^k(G; Z) := H^k(BG; Z).$$

- The **k th characteristic map** of E is the map χ_k induced in cohomology in degree k by the classifying map $X \rightarrow BG$ of E ,

$$\chi_k : H_{Gp}^k(G; Z) = H^k(BG; Z) \rightarrow H^k(X; Z).$$

- The **universal characteristic classes** of G in degree k are the elements $c : BG \rightarrow M^k Z$ of the group cohomology,

$$c \in H_{Gp}^k(G; Z).$$

- The **characteristic classes** of E in degree k are the images $c(E)$ of the universal characteristic classes via the characteristic map,

$$c(E) := \chi_k(c) \in H^k(X; Z).$$

Example 2.14. Here are some of the most commonly used characteristic classes [MS74].

- » When $k > 0$, $Z = \mathbb{Z}/2$, and $G = O(n)$, the **(universal) k th Stiefel-Whitney class** is an element $w_k \in H^k(BO(n), \mathbb{Z}/2)$. The **k th Stiefel-Whitney class** $w_k(E)$ of an $O(n)$ -bundle E lives in $H^k(X; \mathbb{Z}/2)$.
- » When $k = 2k'$, $Z = \mathbb{Z}$, and $G = U(n)$, the **(universal) k th Chern class** is an element $c_{k'} \in H^{2k'}(BU(n), \mathbb{Z})$. The **k' th Chern class** $c_{k'}(E)$ of a $U(n)$ -bundle E lives in $H^{2k'}(X; \mathbb{Z})$.
- » When $n := 2n' + 1$, $k' = 0, \dots, n'$, $k := 4k'$, $Z = \mathbb{Z}$, and $G = SO(n)$, the **(universal) k' th Pontryagin class** is an element $p_{k'} \in H^{4k'}(BSO(n), \mathbb{Z})$. The **k' th Pontryagin class** $p_{k'}(E)$ of an $SO(n)$ -bundle E lives in $H^{4k'}(X; \mathbb{Z})$.

» When $n := 2n'$, $k = n'$, $Z = \mathbb{Z}$, and $G = SO(n)$, the **(universal) Euler class** is an element $e \in H^n(BSO(n), \mathbb{Z})$. The **Euler class** $e(E)$ of an $SO(n)$ -bundle E lives in $H^n(X; \mathbb{Z})$.

Remark 2.15. Let X be a pointed topological space, G be a nice topological group or matrix group, Z an abelian countable group and $k > 0$. Then we have the following bijections

$$\begin{aligned} H_{\mathcal{G}p}^k(G; Z) &:= H^k(BG; Z) && \text{Theorem 2.8} \\ &:= [BG, M^k Z]_* \\ &:= \mathcal{T}op_*(BG, M^k Z) / \simeq_* \\ &= \mathcal{T}op_*(BG, BM^{k-1} Z) / \simeq_* && \text{Corollary 2.3(2) and Lemma 2.12} \\ &= \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}BG, M^{k-1} Z) / \equiv_* && \text{Theorem 1.19} \end{aligned}$$

In particular, every characteristic class $c \in H^k(BG; Z)$ in positive degree $k > 0$ is represented by a **cocycle**

$$\gamma : \tilde{\Omega}BG \rightarrow M^{k-1} Z$$

that fits in the following based homotopy commutative diagram

$$\begin{array}{ccc} BG & \xrightarrow{c} & M^k Z \\ \eta_{BG} \downarrow & \simeq_* & \downarrow \simeq_* \\ B\tilde{\Omega}BG & \xrightarrow{B\gamma} & BM^{k-1} Z. \end{array}$$

3. HOMOTOPY FIBER OF CHARACTERISTIC CLASSES

The main goal of this section is to construct a model of the homotopy fiber of an arbitrary characteristic class $c \in H^k(BG; Z)$, namely to prove Theorem 3.1. This model will be used in the next section to characterize characteristic classes as obstructions to certain possible properties of bundles. Proposition 3.2, which is a tool to prove the theorem, will also be used in the proof of Proposition 5.9.

Theorem 3.1. Let G be a nice topological group, A a nice topological group, and $\alpha : G \rightarrow A$ a map of topological groups.

- (1) The pointed space $\alpha_* EG = EG \otimes_G A$ is nice.
- (2) There is a homotopy fiber sequence of pointed spaces

$$\alpha_* EG \longrightarrow BG \xrightarrow{B\alpha} BA.$$

The idea of considering $\alpha_* EG$ is from [Mit11, Proof of Theorem 10.1]. The key to prove Theorem 3.1(1) is the following proposition.

Proposition 3.2. Let F be a nice pointed topological space, X a nice pointed topological space, and E a pointed fiber bundle over X with fiber F . Then E is a nice pointed space.

A **pointed fiber bundle** over a pointed space X with fiber F consists of a continuous map $p : E \rightarrow X$ such that there exists an open cover $\{U\}$ of X for which there are pointed homeomorphisms $E|_U \cong U \times F$ compatible with p . This is a pointed version of the usual definition of a fiber bundle [Ste99].

The proof of the proposition uses certain properties of *compactly generated weakly Hausdorff spaces*, the definition of which we recall below.

Definition 3.3 ([Str]). Let X be a topological space.

- The space X is **weakly Hausdorff** if for every compact Hausdorff space K and continuous map $f : K \rightarrow X$ the image $f(K)$ is closed.
- A function $f : X \rightarrow Y$ between topological spaces is **k -continuous** if for any compact Hausdorff space C the precomposition $f \circ j : C \rightarrow Y$ of f with a continuous map $j : C \rightarrow E$ is continuous [Str, Proposition 1.11]. The space X is **compactly generated** if every k -continuous function $f : X \rightarrow Y$ is continuous.

It was proven by Strickland in [Str] that compactly generated weakly Hausdorff spaces and continuous maps form a category $CGWH$ of *convenient topological spaces*. Namely, the category $CGWH$ contains all CW-complexes, is bicomplete and is cartesian closed.

Remark 3.4. The category $CGWH$ of compactly generated weakly Hausdorff spaces is reflective in the category CG of compactly generated topological spaces, and the category CG is coreflective in the category \mathcal{Top} of all spaces. The left adjoint $h : CG \rightarrow CGWH$ (described in [Str, Proposition 2.22]) and the right adjoint $k : \mathcal{Top} \rightarrow CG$ (defined in [Str, Definition 1.1]) are called ***h-fication*** and ***k-fication*** respectively. These functors help construct (co)limits in $CGWH$.

- (1) Colimits in $CGWH$ are constructed by *h-fying* the colimits in \mathcal{Top} , as proven in [Str, Corollary 2.23]. In formulas,

$$\operatorname{colim}_i^{CGWH} C_i \cong h(\operatorname{colim}_i^{\mathcal{Top}} C_i).$$

In particular, if a colimit in \mathcal{Top} of a diagram of CGWH spaces is a CGWH space, then it coincides with the colimit in $CGWH$.

- (2) Products in $CGWH$ are constructed by *k-fying* the products in \mathcal{Top} , because the inclusion of $CGWH$ in CG is a right adjoint and products in CG are constructed by *k-fying* the products in \mathcal{Top} [Str, Proposition 2.4]. In formulas,

$$C \times_{CGWH} C' \cong k(C \times_{\mathcal{Top}} C').$$

In particular, if a product in \mathcal{Top} of two CGWH spaces is a CGWH space, then it coincides with the product in $CGWH$.

We now prove two closure properties of CGWH spaces.

- Lemma 3.5.** (1) Every open subset of a CW-complex is a CGWH space.
 (2) Every fiber bundle of which the base space is a CW-complex and the fiber is a CW-complex is a CGWH space.

Proof of Lemma 3.5. (1) As CW-complexes are normal spaces [Hat02, Proposition A.3.], they are in particular regular. Whence, by [Mun75, Lemma 2.1(a)], every open neighbourhood of a point contains a closed neighbourhood of the same point. In particular, for every U open subset of a CW-complex, each point of U has a closed neighborhood in U . By [May99, Chapter 5, Problem 1] this is enough for U to be a CGWH space.

- (2) Let E be a bundle over the nice base space X and with fiber the nice space F . We prove that E is Hausdorff. Let e, e' be two distinct points in E . If $\pi(e) \neq \pi(e')$, then there exist U and U' disjoint neighborhoods of $\pi(e)$ and $\pi(e')$ in X . So $E|_U$ and $E|_{U'}$ are disjoint neighborhoods of e and e' . If $\pi(e) = \pi(e')$, then there exist W a trivialization open that contains $\pi(e)$. As $E|_W \cong W \times F$, which is a product of Hausdorff spaces, the points e and e' can be separated in the open $E|_W$ by two disjoint open neighborhoods.

We prove that E is compactly generated. Let $f : E \rightarrow Y$ be a *k*-continuous function between topological spaces. Let e be a point of E , and U a trivialization open containing $\pi(e)$. The space $E|_U \cong U \times F$ is (homeomorphic to) an open subset of $X \times F$, which is a product of countable CW-complexes, and therefore a CW-complex. By (1), $E|_U$ is a CGWH space, and $f|_{E|_U}$ is *k*-continuous, so $f|_{E|_U}$ is continuous. Since the open subsets of the form $E|_U$ form a cover of E , it follows that f is continuous, and E is a CGWH space. \square

Proof of Proposition 3.2. If σ is a closed cell of X and U is a trivializing open subset of X , the following spaces are CGWH:

- the spaces X , F and σ , as they are CW-complexes,
- the space E , as proven in Lemma 3.5(2),
- the spaces $X \times F$ and $\sigma \times F$, as they are products of countable CW-complexes, and therefore CW-complexes by [Hat02, Theorem A.6],

- the spaces U , $U \cap \sigma$, $U \cap \sigma \times F$, and $U \times F$, as they are open subsets of CW-complexes, and therefore CGWH spaces by Lemma 3.5(1), and
- the space $E|_U$, as it is homeomorphic to $U \times F$, and therefore a CW-complex.

By Remark 3.4(1), colimits of diagrams involving these spaces coincide, be they taken in the category \mathcal{Top} of all spaces or in the category $CGWH$ of CGWH spaces.

We prove that the topological space E is a colimit of its restrictions $E|_\sigma$ to closed cells σ of X . If σ varies over the closed cells of X (and inclusions of those) and U varies over the trivialization open subsets of X for E (and inclusion of those), we have the following homeomorphisms

$$\begin{aligned}
E &\cong \operatorname{colim}_U [E|_U] && \{E|_U\} \text{ is an open cover of } E \\
&\cong \operatorname{colim}_U [U \times F] && E \text{ is locally trivial over } U \\
&\cong \operatorname{colim}_U [(\operatorname{colim}_\sigma [U \cap \sigma]) \times F] && X \text{ has the weak topology and } U \text{ is an open subset} \\
&\cong \operatorname{colim}_U [\operatorname{colim}_\sigma [(U \cap \sigma) \times F]] && - \times F \text{ is a left adjoint (in } CGWH) \\
&\cong \operatorname{colim}_\sigma [\operatorname{colim}_U [(U \cap \sigma) \times F]] && \text{Fubini Theorem [ML98, IX.8]} \\
&\cong \operatorname{colim}_\sigma [(\operatorname{colim}_U [U \cap \sigma]) \times F] && - \times F \text{ is left adjoint (in } CGWH) \\
&\cong \operatorname{colim}_\sigma [\sigma \times F] && \{U \cap \sigma\} \text{ is an open cover of } \sigma.
\end{aligned}$$

Now, every piece $\sigma \times F$ is a countable CW-complex, because it is the product of two countable CW-complexes. We therefore conclude that E is a countable CW-complex. The base point is $(x_0, f_0) \in \{x_0\} \times F$, and is a vertex by construction. \square

Proof of Theorem 3.1. (1) The space $\alpha_* EG = EG \otimes_G A$ is an A -bundle over BG . By applying Proposition 3.2 to $\alpha_* EG$, we conclude that $\alpha_* EG$ is a nice pointed space.

- (2) The following is a generalization of the argument used in [Mit11, Proof of Theorem 10.1]. The tensor product $EA \otimes_G EG$, obtaining by coequalizing the action of G on EA and EG , is a fiber bundle over the nice pointed space BA with fiber the space $\alpha_*(EG)$, which is proven to be nice in (1),

$$\alpha_*(EG) := A \otimes_G EG \xrightarrow{\quad} EA \otimes_G EG \twoheadrightarrow BA.$$

By Proposition 3.2, $EA \otimes_G EG$ is a nice pointed topological space. We then consider the map $(EG, E\alpha) : EG \rightarrow EG \times EA$. It is a pointed morphism of right G -spaces (where the action on $EG \times EA$ is on both factors), which are in fact G -bundles. The morphism $(EG, E\alpha)$ induces a pointed map at the level of quotients:

$$\begin{array}{ccc}
EG & \xrightarrow{\quad} & BG \\
(EG, E\alpha) \downarrow & & \downarrow \overline{(EG, E\alpha)} \\
EG \times EA & \xrightarrow{\quad} & EG \otimes_G EA.
\end{array}$$

Moreover $(EG, E\alpha)$ is a pointed weak homotopy equivalence, and restricts to a homeomorphism on the distinguished fibers because of the equivariance condition. By naturality of the homotopy long exact sequence of Theorem 1.9, the induced map $\overline{(EG, E\alpha)}$ is also a pointed weak equivalence. As BG and $EG \otimes_G EA$ are nice pointed spaces, by Proposition 1.15 this map is in fact a based homotopy equivalence

$$\overline{(EG, E\alpha)} : BG \xrightarrow{\simeq_*} EG \otimes_G EA.$$

As the following diagram commutes

$$\begin{array}{ccccc}
\alpha_* EG := A \otimes_G EG & \xrightarrow{\text{fibre}} & EA \otimes_G EG & \xrightarrow{\pi' \circ \text{pr}_1} & BA \\
& & \uparrow \simeq_* & & \parallel \\
& & (E\alpha, EG) & & \\
& & BG & \xrightarrow{B\alpha} & BA,
\end{array}$$

there exists a pointed map $\alpha_* EG \rightarrow BG$ that yields a homotopy fiber sequence of pointed spaces

$$\alpha_*(EG) \longrightarrow BG \xrightarrow{B\alpha} BA.$$

□

4. CHARACTERISTIC CLASSES AS OBSTRUCTIONS

In this section we give a two-fold interpretation of characteristic classes as obstruction, namely Corollaries 4.7 and 4.14.

Remark 4.1. Note that the homotopy fiber $\alpha_* EG$ introduced in Theorem 3.1 is connected if and only if α is π_0 -surjective. Indeed, by Theorem 1.9, there is an exact sequence

$$\pi_0(G) \xrightarrow{\pi_0(\alpha)} \pi_0(A) \longrightarrow \pi_0(\alpha_* EG) \longrightarrow 0.$$

Definition 4.2. Let G be a nice topological group or a matrix group, A a nice topological group, and $\alpha : G \rightarrow A$ a π_0 -surjective map. The **reduction group** of α is the nice topological group

$$\hat{G}(\alpha) := \tilde{\Omega}(\alpha_* EG).$$

Using Theorems 1.28 and 1.27, we can take the adjoint $\tilde{\Omega}(\alpha_* EG) \rightarrow G$ of the pointed map $\alpha_* EG \rightarrow BG$ constructed in the proof of Theorem 3.1, we obtain a map of nice topological groups

$$j : \hat{G}(\alpha) = \tilde{\Omega}(\alpha_* EG) \rightarrow G.$$

Remark 4.3. Let G be a nice topological group or a matrix group, A a nice topological group, and $\alpha : G \rightarrow A$ a π_0 -surjective map. As by Remark 4.1 $\alpha_* EG$ is a connected nice pointed space, the unit of Definition 1.21

$$\eta_{\alpha_* EG} : \alpha_* EG \rightarrow B\tilde{\Omega}(\alpha_* EG) = B\hat{G}(\alpha)$$

is a pointed weak equivalence, and by Proposition 1.15 a based homotopy equivalence. Therefore the homotopy fibre sequence of Theorem 3.1 can be modified to a new homotopy fibre sequence

$$B\hat{G}(\alpha) \xrightarrow{Bj} BG \xrightarrow{B\alpha} BA.$$

Remark 4.4. Let G be a nice topological group or a matrix group, Z an abelian countable group, $k > 0$, and $c \in H^k(BG; Z)$ a universal characteristic class represented by a π_0 -surjective cocycle $\gamma : \tilde{\Omega}BG \rightarrow M^{k-1}Z$ according to Remark 2.15. The **reduction group** of c is the nice topological group

$$\hat{G}(c) := \hat{G}(\gamma) := \tilde{\Omega}(\gamma_* EG).$$

As it fits in a homotopy fiber sequence

$$B\hat{G}(\gamma) \longrightarrow BG \xrightarrow{B\alpha} K(Z; k),$$

the homotopy long exact sequence of Theorem 1.9 yields

$$0 \longrightarrow \pi_{k-1}(\hat{G}(\gamma)) \xrightarrow{\pi_{k-1}(j)} \pi_{k-1}(G) \longrightarrow Z \longrightarrow \pi_{k-2}(\hat{G}(\gamma)) \xrightarrow{\pi_{k-2}(j)} \pi_{k-2}(G) \longrightarrow 0,$$

and

$$\pi_i(j) : \pi_i(\hat{G}(\gamma)) \cong \pi_i(\hat{G}) \text{ for } i \neq k-1, k-2.$$

We see that the homotopy groups of $\hat{G}(\gamma)$ coincide with the homotopy groups of G , except possibly in degree $k-1$ and $k-2$.

In particular, if c is chosen such that the representing cocycle $\gamma : BG \rightarrow M^{k-1}Z$ is a weak equivalence, it is clear that $\hat{G}(\gamma)$ has the same homotopy groups as G , except in degree $k-1$, where the homotopy group is killed. This fact can be used to build a *Whitehead tower* [Whi52] of the topological group G . As every layer of the Whitehead tower is a localization, by [Far96, Chapter 3, Theorem A.1] we know that every layer of the Whitehead tower of an H-space is itself

an H-space. However, by iterating the construction $\hat{G}(\gamma)$ we in fact obtain a tower of topological groups (and homomorphisms).

Given a characteristic class $c \in H^k(BG; Z)$, we described a group $\hat{G}(c)$ of which the classifying space is the homotopy fiber of the cocycle representing c , and we will see later that this property of $\hat{G}(c)$ is crucial to show that c measures obstruction to group reduction to $\hat{G}(c)$. However, for specific choices of G , k , Z , and c , there is often a standard choice for a group with such property. Here are two examples.

- » When $k = 1$, $Z = \mathbb{Z}/2$, $G = O(n)$ and $c = w_1$, it is known that the classifying space of the special orthogonal group $BSO(n)$ of $SO(n)$ fits into a homotopy fiber sequence

$$BSO(n) \longrightarrow BO(n) \xrightarrow{w_1} B\mathbb{Z}/2.$$

- » When $k = 2$, $Z = \mathbb{Z}/2$, $G = SO(n)$, and $c = w_2$, it is known that the classifying space $BSpin(n)$ of the spin group $Spin(n)$ fits into a homotopy fiber sequence

$$BSpin(n) \longrightarrow BSO(n) \xrightarrow{w_2} K(\mathbb{Z}/2; 2).$$

We now clarify how $SO(n)$ and $Spin(n)$ are related to $\hat{G}(w_1)$ and $\hat{G}(w_2)$ respectively.

Remark 4.5. Let $c \in H^k(BG; Z)$ be a π_0 -surjective characteristic class. If a nice pointed space \hat{B} fits into a homotopy fiber sequence

$$\hat{B} \longrightarrow BG \xrightarrow{c} BM^{k-1}Z,$$

and is a model of classifying space for a group \hat{G} , i.e., there exists a contractible \hat{G} -bundle with base \hat{B} , we wish to compare \hat{G} with $\hat{G}(c)$.

Being both homotopy fibers, \hat{B} and $B\hat{G}(c)$ must have the same based homotopy type. We can use the adjoint map $\tilde{\Omega}\hat{B}\hat{G}(\gamma) \rightarrow \hat{G}$ of a based homotopy equivalence $\hat{B}\hat{G}(\gamma) \rightarrow B$ to construct a map of groups

$$\hat{G}(\gamma) := \tilde{\Omega}(\gamma_*EG) \xrightarrow{\tilde{\Omega}(\eta_{\gamma_*EG})} \tilde{\Omega}B\tilde{\Omega}(\gamma_*EG) = \tilde{\Omega}\hat{B}\hat{G}(\gamma) \longrightarrow \hat{G},$$

that is also a weak equivalence. By anticipating the terminology that will be introduced in Section 4, we think of $\hat{G}(\gamma)$ as a *cofibrant replacement* for \hat{G} . This shows that the general construction $\hat{G}(\gamma)$ is coherent with the known examples.

Theorem 4.6. Let G be a nice topological group, X a nice pointed space, E a pointed G -bundle over X classified by a pointed map $t : X \rightarrow BG$, and A a nice topological group. For a π_0 -surjective map of topological groups $\alpha : G \rightarrow A$, the following are equivalent.

- (a) There exists a pointed $\hat{G}(\alpha)$ -bundle \hat{E} over X and a pointed equivalence of G -bundles over X

$$\hat{E} \otimes_{\hat{G}(\alpha)} G = j_*\hat{E} \cong E.$$

- (b) The classifying map $t : X \rightarrow BG$ of E is killed up to homotopy by $B\alpha$, i.e.,

$$B\alpha \circ t \simeq_* \star : X \rightarrow BA.$$

Proving Theorem 4.6 from Theorem 3.1 is a standard argument, that we briefly recall.

Proof. Condition (b) says that t is killed by $B\alpha$, which is equivalent to saying that t lifts up to based homotopy through the homotopy fiber $B\hat{G}(\alpha)$ via a map $\hat{t} : X \rightarrow B\hat{G}(\alpha)$, as displayed

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \hat{t} & \downarrow t & \searrow 0 & \\ B\hat{G}(\alpha) & \xrightarrow{Bj} & BG & \xrightarrow{B\alpha} & BA. \end{array}$$

Finally, if we then let $\hat{E} := \hat{t}^* E \hat{G}(\alpha)$, we have the following pointed equivalences

$$\begin{aligned} \hat{t}^* \hat{E} &\cong j_* \hat{t}^* E \hat{G}(\alpha) \\ &\cong \hat{t}^* j_* E \hat{G}(\alpha) \\ &\cong \hat{t}^* j_* E \hat{G}(\alpha) && \text{Proposition 1.11(c)} \\ &\cong \hat{t}_*(Bj)^* EG && \text{Proposition 1.12} \\ &\cong (Bj \circ \hat{t})^* EG. && \text{Proposition 1.11(a)} \end{aligned}$$

So t is based homotopic to $Bj \circ \hat{t}$ if and only if $\hat{t}^* \hat{E} \cong t^* EG \cong E$, namely Condition (c). \square

We are interested in applying the previous theorem to a homomorphism $\gamma : \tilde{\Omega}BG \rightarrow M^{k-1}Z$ that represents a characteristic class $c \in H^k(X; Z)$, as mentioned in Remark 2.15. The following result shows that the π_0 -surjective characteristic class c measures the obstruction to the reduction to a $\hat{G}(\gamma)$ -bundle.

Corollary 4.7. Let G be a nice topological group or a matrix group, X a nice pointed space, E a pointed G -bundle over X , Z an abelian countable group and $k > 0$. For a universal characteristic class $c \in H^k(BG; Z)$ represented by a π_0 -surjective cocycle $\gamma : \tilde{\Omega}BG \rightarrow M^{k-1}Z$ as in Remark 2.15, the following are equivalent.

- (A) There exists a $\hat{G}(\gamma)$ -bundle \hat{E} over X and a pointed equivalence of G -bundles over X

$$\hat{E} \otimes_{\hat{G}(\gamma)} G = (\epsilon_G \circ j)_* \hat{E} \cong E.$$

- (B) The characteristic class c of E vanishes,

$$c(E) = 0 \in H^k(X; Z).$$

In order to deduce the corollary from Theorem 4.6, we need a new tool: the *cofibrant replacement* of a bundle. We give the definition and then justify the terminology in Digression 4.9.

Definition 4.8. Let G be a nice topological group or a matrix group, X a CW-complex and E a pointed G -bundle over X classified by $t : X \rightarrow BG$.

- The **cofibrant replacement** of G is the nice topological group

$$G^{\text{cof}} := \tilde{\Omega}BG.$$

The counit from Definition 1.21

$$\epsilon_G : G^{\text{cof}} := \tilde{\Omega}BG \rightarrow G$$

is a homomorphism that is by Remark 1.22 a weak equivalence.

- The **cofibrant replacement** of E is the pointed G^{cof} -bundle over X

$$\begin{aligned} E^{\text{cof}} &:= t^* \tilde{P}BG \\ &\cong t^* \eta_{BG}^* EG^{\text{cof}} && \text{Definition 1.21} \\ &\cong (\eta_{BG} \circ t)^* EG^{\text{cof}}. && \text{Proposition 1.11(1)} \end{aligned}$$

The ϵ_G -equivariant morphism $\tilde{P}BG \rightarrow \epsilon_{G*} \tilde{P}BG \cong EG$ of bundles over BG (from Remark 1.22) induces an ϵ_G -equivariant morphism of bundles over X

$$\epsilon_E : E^{\text{cof}} := t^* \tilde{P}BG \rightarrow t^* \epsilon_{G*} \tilde{P}BG \cong t^* EG \cong E.$$

As a consequence of the natural homotopy long exact sequence of Theorem 1.9, this map is a pointed weak equivalence.

Digression 4.9. Given a nice topological group A , one could check that the assignment

$$G \mapsto \mathcal{G}p(\mathcal{T}op)(G, A)/\equiv_*$$

gives a functor from the category \mathcal{G} of nice topological groups with values in abelian groups,

$$\mathcal{G}p(\mathcal{T}op)(-, A)/\equiv_* : \mathcal{G} \rightarrow \mathcal{A}b.$$

However, there is no reason to expect that this functor will be homotopical with respect to homomorphisms that are also weak equivalences. Referring to [Rie14, Section 2.2], the cofibrant replacement introduced in Definition 4.8 plays the role of a *left deformation* for the functor $\mathcal{G}p(\mathcal{T}op)(-, A)/\equiv_*$, which can then be derived. Using Lemma 6.4, we see that the derived functor gives

$$\mathcal{G}p(\mathcal{T}op)(G^{\text{cof}}, A)/\equiv_* \cong \mathcal{T}op_*(BG, MA)/\simeq_* \cong [BG, MA]_* \in \mathcal{A}b.$$

Note that this functor is now homotopical, because weak equivalences of topological groups are sent by B to based homotopy equivalences of pointed spaces, which are then sent by $[-, MA]_*$ to isomorphisms of abelian groups. In particular, when $A = M^k Z$ is a Milgram delooping, by Theorem 2.8 the derived functor yields group cohomology:

$$\mathcal{G}p(\mathcal{T}op)(G^{\text{cof}}, M^k Z)/\equiv_* \cong [BG, M^{k+1} Z]_* \cong H_{\mathcal{G}p}^k(G; Z).$$

Lemma 4.10. Let X be a nice pointed space and G a nice topological group or a matrix group. There are inverse bijections

$$(-)^{\text{cof}} : {}_X \mathcal{B}un \star_G / \cong \xrightleftharpoons{\quad} {}_X \mathcal{B}un \star_{G^{\text{cof}}} / \cong : \epsilon_{G*}.$$

Proof. We prove that $\epsilon_{G*} \circ (-)^{\text{cof}} = \text{id}$. Let E be a pointed G -bundle classified by $t : X \rightarrow BG$. Then we have the following pointed equivalences of G -bundles over X :

$$\begin{aligned} \epsilon_{G*}(E^{\text{cof}}) &= \epsilon_{G*} t^* \tilde{P}BG \\ &\cong t^* \epsilon_{G*} \tilde{P}BG && \text{Proposition 1.11(3)} \\ &\cong t^* EG && \text{Definition 1.21} \\ &\cong E. \end{aligned}$$

We prove that $(-)^{\text{cof}} \circ \epsilon_{G*} = \text{id}$. Let Q be a pointed $\tilde{\Omega}BG$ -bundle classified by $d : X \rightarrow B\tilde{\Omega}BG$. Then, having fixed a based homotopy inverse η_{BG}^{-1} for η_{BG} , the bundle $\epsilon_{G*}Q$ is classified by $\eta_{BG}^{-1} \circ d$, as shown by the following pointed equivalences of pointed $\tilde{\Omega}BG$ -bundles over X :

$$\begin{aligned} \epsilon_{G*}Q &\cong \epsilon_{G*} d^* E\tilde{\Omega}BG \\ &\cong d^* \epsilon_{G*} E\tilde{\Omega}BG \\ &\cong d^* \epsilon_{G*} (\eta_{BG}^{-1})^* \tilde{P}BG \\ &\cong d^* (\eta_{BG}^{-1})^* \epsilon_{G*} \tilde{P}BG \\ &\cong d^* (\eta_{BG}^{-1})^* EG \\ &\cong (\eta_{BG}^{-1} \circ d)^* EG \end{aligned}$$

We then have the following pointed equivalences of pointed G^{cof} -bundles over X :

$$\begin{aligned} (\epsilon_{G*}Q)^{\text{cof}} &\cong (\eta_{BG}^{-1} \circ d)^* \tilde{P}BG \\ &\cong d^* (\eta_{BG}^{-1})^* \tilde{P}BG \\ &\cong d^* E\tilde{\Omega}BG \\ &\cong Q. \end{aligned}$$

□

Lemma 4.11. Let G be a nice topological group or a matrix group, X a CW-complex, E a pointed G -bundle over X classified by $t : X \rightarrow BG$, $k > 0$ and Z an abelian countable group. For a universal characteristic class $c \in H^k(BG; Z)$ represented by a cocycle $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ the following are equivalent.

- (b) The classifying map $t' = \eta_{BG} \circ t : X \rightarrow BG^{\text{cof}}$ of E^{cof} is killed up to based homotopy by $B\gamma : BG^{\text{cof}} \rightarrow BM^{k-1}Z$,

$$B\gamma \circ \eta_{BG} \circ t \simeq_* \star.$$

- (B) The characteristic class $c(E)$ of E vanishes,

Proof. The conditions (b) and (B) give the vanishing of the top and bottom line respectively in the following diagram,

$$\begin{array}{ccccc} X & \xrightarrow{t'} & BG^{\text{cof}} & \xrightarrow{B\gamma} & BM^{k-1}Z \\ \parallel & & \simeq_* \uparrow \eta_{BG} & & \downarrow \simeq_* \\ X & \xrightarrow{t} & BG & \xrightarrow{c} & M^k Z. \end{array}$$

The diagram commutes up to based homotopy. Indeed, the left square commutes because of the definition of E^{cof} , and the second one because of Remark 2.15. \square

Proof of Corollary 4.7. By applying Theorem 4.6 to E^{cof} and the map $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ that represents the universal characteristic class $c \in H^k(BG; Z)$ according to Remark 2.15, we obtain that the following are equivalent.

- (a) There exists a $\hat{G}(\gamma)$ -bundle \hat{E}' over X and a pointed equivalence of G^{cof} -bundles over X

$$j_* \hat{E} \cong E^{\text{cof}}.$$

- (b) The classifying map $X \rightarrow BG^{\text{cof}}$ of E^{cof} is killed up to based homotopy by $B\gamma : BG^{\text{cof}} \rightarrow BM^{k-1}Z$,

$$B\gamma \circ \eta_{BG} \circ t \simeq_* \star.$$

The fact that (a) is equivalent to (A) is a consequence of Lemma 4.10. More explicitly, (a) is obtained from (A) by taking the cofibrant replacement, and (A) is obtained from (a) by applying ϵ_{G*} . By Lemma 4.11, (b) is equivalent to (B). \square

Example 4.12. Thanks to Lemma 4.10 and Remark 4.5, Corollary 4.7 recovers some classical results [MS74]. We here mention two of them.

- » The first Stiefel-Whitney class $w_1(E)$ of an $O(n)$ -bundle E vanishes if and only if there exists an $SO(n)$ -bundle \hat{E} such and a pointed equivalence of $O(n)$ -bundles over X

$$E \cong \hat{E} \otimes_{SO(n)} O(n),$$

namely the bundle is orientable.

- » The second Stiefel-Whitney class $w_2(E)$ of an $SO(n)$ -bundle E vanishes if and only if there exists a $Spin(n)$ -bundle \hat{E} and a pointed equivalence of $SO(n)$ -bundles over X

$$E \cong \hat{E} \otimes_{Spin(n)} SO(n),$$

namely the bundle admits a spin structure.

We now prove that characteristic classes measure obstruction to the existence of an equivariant extension of the universal cocycle.

Theorem 4.13. Let G be a nice topological group, X a pointed CW-complex, E a pointed G -bundle over X classified by $t : X \rightarrow BG$, and A a nice topological group. For a map of topological groups $\alpha : G \rightarrow A$ the following are equivalent.

- (b) The map $t : X \rightarrow BG$ is killed up to homotopy by $B\alpha$, i.e.,

$$B\alpha \circ t \simeq_* \star : X \rightarrow BA.$$

- (c) The map $\alpha : G \rightarrow A$ extends to an α -equivariant pointed map $E \rightarrow A$.

$$\begin{array}{ccc} G & \xrightarrow{\quad} & E \\ \alpha \downarrow & \swarrow & \\ A & & \end{array}$$

- (d) The bundle $\alpha_* EG$ is trivial, i.e.,

$$\alpha_* E \cong X \times A.$$

Proof. We prove that $[(b) \Leftrightarrow (d)]$. There are pointed equivalences of pointed A -bundles over X :

$$\begin{aligned} (Ba \circ t)^* EA &\cong t^*(Ba)^* EA \\ &\cong t^* a_* EG \\ &\cong a_* t^* EG \\ &\cong a_* E. \end{aligned}$$

Therefore, by Theorem 1.19, the map $Ba \circ t$ is based homotopic to the pointed constant map if and only if $a_* E$ is pointed equivalent to the trivial bundle.

We prove that $[(c) \Rightarrow (d)]$. If there exists an α -equivariant pointed map $\kappa : E \rightarrow A$, then $(\pi, \kappa) : E \rightarrow X \times A$ is a pointed α -equivariant map over X , and it induces by Proposition 1.12 a pointed equivalence of pointed A -bundles over X

$$\overline{(\pi, \kappa)} : \alpha_* E \cong X \times A.$$

We prove that $[(d) \Rightarrow (c)]$. Given a pointed equivalence $\phi : \alpha_* E \cong X \times A$ of pointed A -bundles over X , we use it to construct the α -equivariant pointed map

$$E \xrightarrow{\eta_a} \alpha_* E \xrightarrow{\phi} X \times A \xrightarrow{\text{pr}_2} A,$$

which therefore extends α . □

Again, we are interested in applying the previous theorem to a homomorphism $\gamma : \tilde{\Omega}BG \rightarrow M^{k-1}Z$ that represents a universal characteristic class $c \in H^k(BG; Z)$, as mentioned in Remark 2.15. The following result shows that the characteristic class c measures the obstruction to the existence of an equivariant extension of γ to E^{cof} .

Corollary 4.14. Let G be a nice topological groups or a matrix group, X a CW-complex, E a pointed G -bundle over X , $k > 0$ and Z an abelian countable group. For a universal characteristic class $c \in H^k(BG; Z)$ represented by $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ the following are equivalent.

(B) The characteristic class $c(E)$ of E vanishes,

$$c(E) = 0 \in H^k(X; Z).$$

(C) The map $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ extends to a γ -equivariant pointed map $E^{\text{cof}} \rightarrow M^{k-1}Z$.

Proof. By applying Theorem 4.13 to E^{cof} and the cocycle $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ that represents the universal characteristic class $c \in H^k(BG; Z)$, we obtain that the following are equivalent.

(b) The classifying map $X \rightarrow BG^{\text{cof}}$ of E^{cof} is killed up to based homotopy by $B\gamma : BG^{\text{cof}} \rightarrow BM^{k-1}Z$,

$$B\gamma \circ \eta_{BG} \circ t \simeq_* \star.$$

(C) The map $\gamma : G^{\text{cof}} \rightarrow M^{k-1}Z$ extends to a γ -equivariant pointed map $E^{\text{cof}} \rightarrow M^{k-1}Z$.

By Lemma 4.11, (b) is equivalent to (B). □

Example 4.15. Corollary 4.14 recover some classical results [MS74]. We deduce, for instance, that the first Stiefel-Whitney class $w_1(E)$ of an $O(n)$ -bundle E vanishes if and only if there exists an equivariant function $E \rightarrow \mathbb{Z}/2$, that corresponds to a coherent choice of sign for the bundle E , namely an orientation.

Indeed, by Corollary 4.14, the vanishing of $w_1(E)$ yields an equivariant map $E^{\text{cof}} \rightarrow \mathbb{Z}/2$. Then, the topology of $\mathbb{Z}/2$ is discrete, and therefore the map factors through $\pi_0(E^{\text{cof}}) \cong \pi_0(E)$, as displayed:

$$\begin{array}{ccccc} G^{\text{cof}} & \xrightarrow{c} & E^{\text{cof}} & \xrightarrow{\quad} & \mathbb{Z}/2 \\ \epsilon_G \downarrow & & \epsilon_E \downarrow & \searrow & \uparrow \\ G & \xrightarrow{c} & E & \xrightarrow{\quad} & \pi_0(E) \\ & & & \downarrow \cong & \\ & & & \pi_0(E^{\text{cof}}) & \end{array}$$

(A dashed arrow also goes from $\pi_0(E)$ to $\mathbb{Z}/2$.)

In particular, there exists an orientation $E \rightarrow \mathbb{Z}/2$, as desired.

5. THE CATEGORY OF POINTED BUNDLES

This section provides the framework for the sequence of abelian groups that will be constructed in Section 6, and proven to be exact in Section 7.

Definition 5.1. We denote by $\mathcal{Bun}\star$ the **category of pointed principal bundles**.

- An object consists of a triple (X, G, E) , where X is a pointed topological space, G is a topological group and E is a pointed G -bundle over X . The space X is the **base space**, the group G is the **structure group** and the space E is the **total space**. We denote this object by ${}_XE_G$, or just by E when there is no confusion.
- A morphism ${}_XE_G \rightarrow {}_{X'}E'_{G'}$ consists of a triple (ψ, g, a) , where $g : X \rightarrow X'$ is a pointed map, $a : G \rightarrow G'$ is a homomorphism of topological groups, and $\psi : E \rightarrow E'$ is a pointed map that is a -equivariant and g -coequivariant. The pointed map g is called the **geometric information**, the homomorphism a is called **algebraic information**, and the pointed map ψ is called **total information**. We denote this object by ${}_g\psi_a$, or just ψ when there is no confusion.
- Composition and identity are defined in the obvious way.

Remark 5.2. Restricting ourselves to *pointed* bundles allows a better interaction between the category of bundles and the categories of groups and spaces we work with. Here are a few properties that exploit the pointed framework.

- For every pointed bundle ${}_XE_G$, the inclusion ι of the distinguished fibre and the projection π on the base space are morphisms in $\mathcal{Bun}\star$, as displayed

$${}_*\mathcal{G}_G \xrightarrow{{}_*\iota_G} {}_XE_G \xrightarrow{{}_X\pi_*} {}_XX_*.$$

- Every morphism ${}_X\psi_a : {}_XE_G \rightarrow {}_{X'}E'_{G'}$, induces a on the distinguished fibers and g on the base spaces, namely, there is a commutative diagram in $\mathcal{Bun}\star$:

$$\begin{array}{ccccc} {}_*\mathcal{G}_G & \xrightarrow{{}_*\iota_G} & {}_XE_G & \xrightarrow{{}_X\pi_*} & {}_XX_* \\ {}_*a_a \downarrow & & {}_g\psi_a \downarrow & & {}_gg_* \downarrow \\ {}_*\mathcal{G}'_{G'} & \xrightarrow{{}_*\iota'_{G'}} & {}_{X'}E'_{G'} & \xrightarrow{{}_{X'}\pi'_*} & {}_{X'}X'_* \end{array}$$

- Pointed equivalences are isomorphisms in $\mathcal{Bun}\star$.
- Topological groups and pointed spaces embed fully faithfully into pointed bundles, as stated more precisely in Proposition 5.3.

Proposition 5.3. (1) The assignment

$$X \mapsto {}_XX_*,$$

that interprets a pointed space as a **discrete bundle** over itself with trivial structure group, gives a fully faithful inclusion

$$\mathcal{Top}_\star \hookrightarrow \mathcal{Bun}\star$$

of pointed spaces in pointed principal bundles.

(2) The assignment

$$G \mapsto {}_*\mathcal{G}_G,$$

that interprets a topological group as a **codiscrete G -bundle** over a point, gives a fully faithful inclusion

$$\mathcal{Gp}(\mathcal{Top}) \hookrightarrow \mathcal{Bun}\star$$

of topological groups in pointed principal bundles. \square

Every topological abelian group, seen as codiscrete bundle, is monoid in $\mathcal{Bun}\star$ with respect to the product. As a consequence, the contravariant functor represented by a topological abelian group takes values into abelian groups.

Proposition 5.4. Let A be a topological abelian group. Then the assignment

$${}_X E_G \mapsto \mathcal{Bun}\star({}_X E_G, {}_\star A_A) =: \mathcal{Bun}\star(E, A)$$

defines a contravariant functor in abelian groups

$$\mathcal{Bun}\star(-, A) : \mathcal{Bun}\star^{\text{op}} \rightarrow \mathcal{Ab},$$

where the group structure is given by pointwise multiplication. \square

Remark 5.5. Let E be a pointed G -bundle over X , with X a nice pointed space and G a nice topological group or a matrix group. By Theorems 1.26 and 1.27, there exist classifying maps

$$t : X \rightarrow BG \text{ and } b : \tilde{\Omega}X \rightarrow G$$

that yield pointed equivalences of G -bundle over X

$$t^\star EG \cong E \cong b_\star \tilde{P}X.$$

Having chosen two such maps, we can build the **Puppe sequence** of the bundle

$$\tilde{\Omega}X \xrightarrow{b} G \xrightarrow{\iota} E \xrightarrow{\pi} X \xrightarrow{t} BG$$

which, thanks to Proposition 5.3 and Remark 5.2(1), lives in the category $\mathcal{Bun}\star$ as

$$\star \tilde{\Omega}X_{\tilde{\Omega}X} \xrightarrow{\star b_b} \star G_G \xrightarrow{\star \iota_G} {}_X E_G \xrightarrow{X \pi_\star} {}_X X_\star \xrightarrow{t t_\star} {}_{BG} BG_\star.$$

Let now A be a topological abelian group. By applying the functor $\mathcal{Bun}\star(-, A)$ of Proposition 5.4 to the Puppe sequence of Remark 5.5, we obtain a diagram of abelian groups,

$$\mathcal{Bun}\star(BG, A) \xrightarrow{t^\star} \mathcal{Bun}\star(X, A) \xrightarrow{\pi_\star} \mathcal{Bun}\star(E, A) \xrightarrow{\iota^\star} \mathcal{Bun}\star(G, A) \xrightarrow{b^\star} \mathcal{Bun}\star(\tilde{\Omega}X, A),$$

which in general will not be exact, and not even a complex of abelian groups.

We will define an equivalence relation \simeq_+ on $\mathcal{Bun}\star(E, A)$ with the following properties.

- The **plus-homgroup** $[E, A]_+$, defined as the quotient

$$[E, A]_+ := \mathcal{Bun}\star(E, A) / \simeq_+,$$

is an abelian group with respect to pointwise multiplication.

- When $A = M^k Z = K(Z; k)$ is a k -fold Milgram delooping and $E = X$ is a discrete bundle or $E = G^{\text{cof}}$ is a cofibrant codiscrete bundle, the plus-homgroup recovers ordinary (group) cohomology

$$[X, M^k Z]_+ \cong H^k(X; Z) \quad \text{and} \quad [G^{\text{cof}}, A]_+ \cong H^k(BG; Z).$$

- There is an exact sequence of abelian groups obtained by quotienting the diagram above,

$$(*) \quad [BG, A]_+ \xrightarrow{t^\star} [X, A]_+ \xrightarrow{\pi_\star} [E, A]_+ \xrightarrow{\iota^\star} [G, A]_+ \xrightarrow{b^\star} [\tilde{\Omega}X, A]_+.$$

Assuming these requirements for the equivalence relation \simeq_+ , it is possible to identify the generic representatives in $\mathcal{Bun}\star(E, A)$ of elements in the image of π_\star , and certain elements in $\mathcal{Bun}\star(E, A)$ that must be killed in $[E, A]_+$. We refer to these two classes of morphisms as *of type i* and *type k*, in Definition 5.8 below. In order to define the morphisms of type k, we need to introduce some new terminology.

Remark 5.6. We recall from Definition 1.25 that a universal bundle is a contractible pointed bundle over a CW-complex. Here are some models of universal bundles and classifying spaces.

- » If G is a nice topological group, $E^w G$ is a universal bundle.
- » If G is a matrix group, $St(G)$ is a universal bundle.
- » If $G = \star$ is the trivial group, any universal bundle is a contractible nice pointed space that coincides with the classifying space.

Remark 5.7. Let G be a topological group, and \tilde{E} a universal bundle for G with base space \tilde{B} . If then E is a pointed G -bundle over a CW-complex X pointed in a vertex, there exists a classifying map $t : X \rightarrow \tilde{B}$ such that $t^*\tilde{E} \cong E$. In fact, t determines further data, because the following data are equivalent.

- (1) a pointed map $t : X \rightarrow \tilde{B}$ together with a pointed equivalence of G -bundle over X

$$t^*\tilde{E} \cong E;$$

- (2) a pointed G -equivariant map

$$T : E \rightarrow EG;$$

- (3) a pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi} & X \\ T \downarrow & \lrcorner & t \downarrow \\ \tilde{E} & \xrightarrow{\tilde{\pi}} & \tilde{B}. \end{array}$$

In this case, the map $T : E \rightarrow \tilde{E}$ is a **universal map** for E and the map $t : X \rightarrow \tilde{B}$ is a **classifying map** for E . Note that both maps live in $\mathcal{Bun}\star$, as displayed

$${}_tT_G : {}_XE_G \rightarrow {}_{\tilde{B}}\tilde{E}_G \text{ and } {}_tt_\star : {}_{\tilde{B}}\tilde{B}_\star \rightarrow {}_XX_\star.$$

Note that any two classifying space have the same based homotopy type.

Definition 5.8. Let X be a CW-complex pointed in a vertex, G a nice topological group or a matrix group, E a pointed G -bundle over X , and A a topological abelian group.

- A morphism $\psi : E \rightarrow A$ in $\mathcal{Bun}\star$ is **of type k** if it factors through a universal map $T : E \rightarrow \tilde{E}$ in $\mathcal{Bun}\star$ for \tilde{E} a nice pointed universal bundle. Explicitly, this means that ψ factors as

$$\star\psi_a : {}_XE_G \xrightarrow{{}_tT_G} {}_{\tilde{B}}\tilde{E}_G \xrightarrow{\star\kappa_a} \star A_A$$

for some $\kappa : \tilde{E} \rightarrow A$ in $\mathcal{Bun}\star$. We denote by \mathcal{M}_k the class of morphisms of type k in $\mathcal{Bun}\star(E, A)$.

- A morphism $\psi : E \rightarrow A$ in $\mathcal{Bun}\star$ is **of type i** if it factors through the projection $\pi : E \rightarrow X$ in $\mathcal{Bun}\star$, or equivalently if the algebraic information is trivial, i.e., $a = e_A$. Explicitly, this means that ψ factors as

$$\star\psi_\star : {}_XE_G \xrightarrow{\pi\pi_\star} {}_XX_\star \xrightarrow{\star s_\star} \star A_A$$

for some $s : X \rightarrow A$ in $\mathcal{Bun}\star$. We denote by \mathcal{M}_i the class of morphisms of type i in $\mathcal{Bun}\star(E, A)$.

- A morphism $\psi : E \rightarrow A$ in \mathcal{Bun} is **of type ik** if it factors through a classifying space \tilde{B} via $\pi \circ T = t \circ \pi : E \rightarrow \tilde{B}$ in $\mathcal{Bun}\star$, or equivalently if it is of type i and of type k (see Proposition 5.9(4)). Explicitly, this means that ψ factors as

$$\begin{array}{ccccc} \star\psi_\star : {}_X\tilde{E} & \xrightarrow{\pi\pi_\star} & {}_XX_\star & \xrightarrow{{}_tt_\star} & {}_{\tilde{B}}\tilde{B}_\star & \xrightarrow{\star s'_\star} & \star A_A \\ \parallel & & & & \parallel & & \parallel \\ {}_X\tilde{E} & \xrightarrow{{}_tT_G} & {}_{\tilde{B}}\tilde{E}_G & \xrightarrow{\pi'\pi'_\star} & {}_{\tilde{B}}\tilde{B}_\star & \xrightarrow{\star s'_\star} & \star A_A \end{array}$$

for some $s' : \tilde{B} \rightarrow A$ in $\mathcal{Bun}\star$. We denote by \mathcal{M}_{ik} the class of morphisms of type ik in $\mathcal{Bun}\star(E, A)$.

- A morphism $\psi : E \rightarrow A$ in \mathcal{Bun} is **of type w** if it is of type i, and the factor $s : X \rightarrow A$ is based homotopic to a map that factors through a classifying space \tilde{B} in \mathcal{Top}_* . Explicitly, this means that ψ factors (up to based homotopy) as

$$\begin{array}{ccccc} \star\psi_\star : X\tilde{E} & \xrightarrow{X\pi_\star} & X X_\star & \xrightarrow{\star s_\star} & \star A_\star \\ \parallel & & \parallel & \simeq_\star & \downarrow \\ X\tilde{E} & \xrightarrow{\pi\pi_\star} & X X_\star & \xrightarrow{t_\star} & \tilde{B}\tilde{B}_\star \xrightarrow{\star s'_\star} \star A_A \end{array}$$

for some $s : X \rightarrow A$ and $s' : \tilde{B} \rightarrow A$. We denote by \mathcal{M}_w the class of morphisms of type w in $\mathcal{Bun}\star(E, A)$.

The reason for the terminology is that, referring to the diagram (*), morphisms of type i will lie in the *image* of π_\star , morphisms of type k will be *killed* in $[E, A]_+$, and the class of morphisms of type w includes the class of morphisms of type ik. In this sense, the condition w is *weaker* than the condition ik. In the end, we shall kill all morphisms of type k and of type w.

We now prove useful properties of these distinguished classes of maps. Recall that the dot "." denotes pointwise multiplication.

Proposition 5.9. Let X be a CW-complex pointed in a vertex, G a nice topological group or a matrix group, E a pointed G -bundle over X and A an abelian topological group.

- (1) For ψ and ψ' in $\mathcal{Bun}\star(E, A)$, if ψ and ψ' are of type k, then $\psi \cdot \psi'$ is of type k. Equivalently,

$$\mathcal{M}_k \cdot \mathcal{M}_k \subset \mathcal{M}_k \subset \mathcal{Bun}\star(E, A).$$

- (2) For ψ and ψ' in $\mathcal{Bun}\star(E, A)$, if ψ and ψ' are of type w, then $\psi \cdot \psi'$ is of type w. Equivalently,

$$\mathcal{M}_w \cdot \mathcal{M}_w \subset \mathcal{M}_w \subset \mathcal{Bun}\star(E, A).$$

- (3) For ψ in $\mathcal{Bun}\star(E, A)$, if ψ is of type ik, then ψ is of type w. Equivalently,

$$\mathcal{M}_{ik} \subset \mathcal{M}_w \subset \mathcal{Bun}\star(E, A).$$

- (4) For ψ in $\mathcal{Bun}\star(E, A)$, ψ is of type ik if and only if it is of type i and of type k. Equivalently,

$$\mathcal{M}_{ik} = \mathcal{M}_i \cap \mathcal{M}_k \subset \mathcal{Bun}\star(E, A).$$

Proof. (1) As ψ and ψ' are of type k, they factor through universal bundles \tilde{E} and \tilde{E}' , as displayed:

$$\psi : E \xrightarrow{T} \tilde{E} \xrightarrow{\kappa} A \quad \text{and} \quad \psi' : E \xrightarrow{T'} \tilde{E}' \xrightarrow{\kappa'} A.$$

Then the product $\psi \cdot \psi'$ factors through $\tilde{E} \times \tilde{E}'$, as displayed

$$\psi \cdot \psi' : E \xrightarrow{(T, T')} \tilde{E} \times \tilde{E}' \xrightarrow{\kappa \times \kappa'} A \times A \xrightarrow{\mu} A.$$

We note that the pointed space $\tilde{E} \times \tilde{E}'$ is a universal bundle for G . Indeed, it is contractible, and is a pointed G -bundle over $\tilde{E} \otimes_G \tilde{E}'$, which is a bundle over \tilde{B} with fiber \tilde{E}' . Then $\tilde{E} \otimes_G \tilde{E}'$ is nice by Lemma 3.2, and $\tilde{E} \times \tilde{E}'$ is nice because it is a product of countable CW-complexes.

- (2) As ψ and ψ' are of type w, they factor through the base space X , as displayed:

$$\psi : E \xrightarrow{\pi} X \xrightarrow{s} A \quad \text{and} \quad \psi' : E \xrightarrow{\pi} X \xrightarrow{s'} A.$$

Then the product $\psi \cdot \psi'$ factors through X , as displayed

$$\psi \cdot \psi' : E \xrightarrow{\pi} X \xrightarrow{(s, s')} A \times A \xrightarrow{\mu} A.$$

Moreover, s and s' factor up to based homotopy through some classifying space \tilde{B} , as displayed:

$$s : X \xrightarrow{t} \tilde{B} \xrightarrow{\bar{s}} A \quad \text{and} \quad s' : X \xrightarrow{t'} \tilde{B}' \xrightarrow{\bar{s}'} A.$$

As \tilde{E} is a G -bundle and \tilde{E}' is a universal bundle for G , there exists a pointed map $\tau : \tilde{B} \rightarrow \tilde{B}'$ such that

$$\tau^* \tilde{E}' \cong \tilde{E}.$$

We then have the following pointed equivalences of G -bundles over X :

$$\begin{aligned} (\tau \circ t)^* \tilde{E}' &\cong t^* \tau^* \tilde{E}' \\ &\cong t^* \tilde{E} \\ &\cong E \\ &\cong t'^* \tilde{E}', \end{aligned}$$

which implies by Theorem 1.19 that

$$t' \simeq_* \tau \circ t.$$

In other words, $s' : X \rightarrow A$ also factors through \tilde{B} up to based homotopy,

$$s : X \xrightarrow{t} \tilde{B} \xrightarrow{\bar{s}} A \text{ and } s' : X \xrightarrow{t} \tilde{B} \xrightarrow{\tau} \tilde{B}' \xrightarrow{\bar{s}'} A.$$

Then the product $s \cdot s'$ factors up to based homotopy through \tilde{B} , as displayed

$$s \cdot s' : X \xrightarrow{t} \tilde{B} \xrightarrow{(\bar{s}, \bar{s}' \circ \tau)} A \times A \xrightarrow{\mu} A.$$

It follows that $\psi \cdot \psi'$ is of type w.

- (3) This is clear from the definitions.
- (4) The direction $[\implies]$ is clear. We prove the direction $[\impliedby]$. As ψ is of type k, ψ factors through a universal bundle \tilde{E} as displayed

$$\psi : E \xrightarrow{T} \tilde{E} \xrightarrow{\kappa} A$$

Then, as ψ is of type i, the algebraic information $a(\kappa)$ of κ vanishes:

$$0 = a(\psi) = a(\kappa \circ T) = a(\kappa) \circ a(T) = a(\kappa) \circ \text{id}_G = a(\kappa).$$

So the map κ is of type i, and factors through the quotient $\tilde{E}/G = \tilde{B}$. Thus we can write ψ as follows

$$\begin{array}{ccccc} E & \xrightarrow{\psi} & A & & \\ \parallel & & \parallel & & \\ E & \xrightarrow{T} \tilde{E} & \xrightarrow{\kappa} & A & \\ \parallel & & \parallel & & \\ E & \xrightarrow{T} \tilde{E} & \twoheadrightarrow \tilde{B} & \xrightarrow{s'} & A, \\ \downarrow & & \downarrow & \nearrow & \\ X & \xrightarrow{t} & \tilde{B} & & \end{array}$$

and see that it factors through the classifying space \tilde{B} . It follows that ψ is of type ik. \square

Remark 5.10. Every map of type k factors through a contractible space, and every map of type w is homotopic to a map of type ik, which factors through a contractible space. It follows that maps of type k and w are based homotopic to constant maps.

6. PLUS-COHOMOLOGY GROUPS OF PRINCIPAL BUNDLES

Definition 6.1. Let X be CW-complex pointed in a vertex, G a nice topological group or a matrix group, E a pointed G -bundle over X and A an abelian topological group.

- An **ik-decomposition** of $\psi : E \rightarrow A$ in $\mathcal{Bun}\star$ consists of a morphism $\psi_i : E \rightarrow A$ of type i and a morphism $\psi_k : E \rightarrow A$ of type k such that

$$\psi = \psi_i \cdot \psi_k \in \mathcal{Bun}\star(E, A).$$

- A **wk-decomposition** of $\psi : E \rightarrow A$ in $\mathcal{Bun}\star$ consists of a morphism $\psi_w : E \rightarrow A$ of type w and a morphism $\psi_k : E \rightarrow A$ of type k such that

$$\psi = \psi_w \cdot \psi_k \in \mathcal{Bun}\star(E, A).$$

- Two morphisms $\psi, \psi' : E \rightarrow A$ in $\mathcal{Bun}\star$ are **plus-equivalent**, and we write $\psi \simeq_+ \psi'$, if and only if $\psi \cdot \psi'^{-1} : E \rightarrow A$ admits a wk-decomposition

$$\psi \cdot \psi'^{-1} = \psi_w \cdot \psi_k \in \mathcal{Bun}\star(E, A).$$

The relation \simeq_+ is an equivalence relation. Indeed, it is easily verified to be reflexive and symmetric, and is transitive as a consequence of Proposition 5.9 (1) and (2). It is also a congruence with respect to pointwise multiplication, again thanks to Proposition 5.9 (1) and (2). We denote by $[\psi]_+$ the class of $\psi : E \rightarrow A$ with respect to the relation \simeq_+ , and by $[E, A]_+$ the quotient

$$[E, A]_+ := \mathcal{Bun}(E, A) / \simeq_+ \in \mathcal{Ab},$$

which is an abelian group with respect to pointwise multiplication. If Z is an abelian countable group and $k \geq 0$, we define the **plus-cohomology group** $H_+^k(E; Z)$ by

$$H_+^k(E; Z) := [E^{\text{cof}}, M^k Z]_+.$$

When $E = X$ is a discrete bundle, or $E = G^{\text{cof}}$ is a cofibrant codiscrete bundle, the plus-cohomology group recovers ordinary (group) cohomology, as we show below, in Corollary 6.5.

Recall that, if A is a topological group, the identity of A is e_A . We also denote by e_A any constant map into A with value e_A .

Proposition 6.2. Let A be an abelian nice topological group.

- (1) For any CW-complex X pointed in a vertex, two pointed maps $s, s' : X \rightarrow A$ are plus equivalent if and only if they are based homotopic, i.e.,

$$s \simeq_+ s' : X \rightarrow A \text{ if and only if } s \simeq_* s' : X \rightarrow A.$$

- (2) For any nice topological group G , two maps of topological groups $a, a' : G \rightarrow A$ are plus-equivalent if and only if they are based algebraically equivalent, i.e.,

$$a \simeq_+ a' : G \rightarrow A \text{ if and only if } a \equiv_* a' : G \rightarrow A.$$

The following lemma is preliminary to Proposition 6.2, but also to Lemma 7.2, which is used in the next section.

Lemma 6.3. Let G be a nice topological group, and A a nice topological group. For a homomorphism $a : G \rightarrow A$ the following are equivalent.

- (1) The homomorphism a extends to an a -equivariant pointed map

$$\kappa : EG \rightarrow A \text{ in } \mathcal{Bun}\star.$$

- (2) There is a pointed equivalence of A -bundles over BG

$$a_* EG \cong BG \times A.$$

- (3) There is a pointed equivalence of A -bundles over BG

$$(a \circ \epsilon_G)_* \tilde{P}BG \cong BG \times A.$$

(4) The homomorphism a is based equivalent to the constant homomorphism,

$$a \equiv_{\star} e_A : G \rightarrow A.$$

Proof of Lemma 6.3. Recall that every pointed A -equivariant map between A -bundles over BG that is compatible with the projection onto BG is a pointed equivalence of A -bundles over BG .

We prove $[(1) \Rightarrow (2)]$. By Proposition 1.12, the map κ factors through $a_{\star}EG$, as displayed

$$\kappa : EG \xrightarrow{\eta_a} a_{\star}EG \xrightarrow{\bar{\kappa}} A$$

Moreover, the projection of $a_{\star}EG$ gives a map

$$\pi : a_{\star}EG \longrightarrow BG.$$

Combining π and $\bar{\kappa}$, we obtain a pointed equivalence of A -bundles over BG

$$(\pi, \bar{\kappa}) : a_{\star}EG \xrightarrow{\cong} BG \times A.$$

We prove $[(1) \Leftarrow (2)]$. Given a pointed equivalence of A -bundle over BG

$$\phi : a_{\star}EG \xrightarrow{\cong} BG \times A,$$

the following a -equivariant map κ extends the map a :

$$\kappa : EG \xrightarrow{\eta_a} a_{\star}EG \xrightarrow[\cong]{\phi} BG \times A \xrightarrow{\text{pr}_2} A.$$

We prove $[(2) \Leftrightarrow (3)]$. This is a consequence of the fact that, by Theorem 1.19 and Definition 1.21 there is a pointed equivalence of G -bundles over BG

$$\epsilon_{G\star} \tilde{P}BG \cong EG.$$

We prove $[(3) \Leftrightarrow (4)]$. As $\tilde{P}BG$ classifies $\tilde{\Omega}BG$ -bundles, by Theorem 1.19, we have that

$$(a \circ \epsilon_G)_{\star} \tilde{P}BG \cong BG \times A$$

if and only if

$$a \circ \epsilon_G \equiv_{\star} e_A \quad \text{namely} \quad Ba \circ B\epsilon_G = B(a \circ \epsilon_G) \simeq_{\star} \star.$$

Now, as ϵ_G and $E\epsilon_G$ are weak equivalences, by Proposition 1.12(2) the map $B\epsilon_G$ is a pointed weak equivalence between nice pointed spaces and therefore by Remark 1.15 a based homotopy equivalence. So the condition can be rewritten as

$$Ba \simeq_{\star} \star \quad \text{namely} \quad a \equiv_{\star} e_A.$$

□

Proof of Proposition 6.2. (1) We prove the implication $[\Leftarrow]$. If $s \simeq_{\star} s' : X \rightarrow A$, then $s' \cdot s^{-1} \simeq_{\star} e_A : X \rightarrow A$, and finally $s' \cdot s^{-1}$ is of type w in $\mathcal{B}un\star(X, A)$. So $s \simeq_{+} s'$.

We prove the implication $[\Rightarrow]$. If $s \simeq_{+} s' : X \rightarrow A$, then $s' \cdot s^{-1} = \psi_k \cdot \psi_w$ where $\psi_k : X \rightarrow A$ is of type k and $\psi_w : X \rightarrow A$ is of type w in $\mathcal{B}un\star(X, A)$. By Remark 5.10, all maps of type k and all maps of type w are based homotopic to the constant pointed map, and therefore

$$s' \cdot s^{-1} = \psi_k \cdot \psi_w \simeq_{\star} e_A \cdot e_A = e_A : X \rightarrow A.$$

It follows that $s \simeq_{\star} s'$.

(2) We prove the implication $[\Leftarrow]$. If $a \equiv_{\star} a' : G \rightarrow A$, then $a' \cdot a^{-1} \equiv_{\star} e_A : G \rightarrow A$. By Lemma 6.3, $a' \cdot a^{-1}$ factors through EG , and is therefore of type k in $\mathcal{B}un\star(G, A)$. It follows that $a \simeq_{+} a'$.

We prove the implication $[\Rightarrow]$. If $a \simeq_{+} a' : G \rightarrow A$, then $a' \cdot a^{-1} = \psi_k \cdot \psi_w$ where $\psi_k : G \rightarrow A$ is of type k and $\psi_w : G \rightarrow A$ is of type w in $\mathcal{B}un\star(G, A)$. As ψ_k is of type k, it factors through some classifying space \tilde{E} for G . As EG is a G -bundle and \tilde{E} is a universal

bundle for G , there exists a G -equivariant pointed map $EG \rightarrow \tilde{E}$. Then ψ_k also factors through EG , as displayed

$$\begin{array}{ccccc} \psi_k : G & \hookrightarrow & \tilde{E} & \xrightarrow{\tilde{\kappa}} & A. \\ \parallel & & \uparrow & & \\ G & \hookrightarrow & EG & & \end{array}$$

By Lemma 6.3, ψ_k is then based algebraic equivalent to the constant pointed map. On the other hand, ψ_w is constant because it factors through the base space of G , which is trivial. Therefore

$$a' \cdot a^{-1} = \psi_k \cdot \psi_w = \psi_k \cdot \star = \psi_k \equiv_\star \star : G \rightarrow A.$$

It follows that $a \equiv_\star a'$.

□

Lemma 6.4. Let A be an abelian nice topological group. For every nice pointed space X there is a natural isomorphism of abelian groups

$$[\tilde{\Omega}X, A]_+ \xrightarrow{\cong} [X, MA]_+ \in \mathcal{Ab}.$$

Proof. Let $\tau : BA \rightarrow MA$ be the homotopy natural based homotopy equivalence of Corollary 2.3(4). We claim that, for every X , the following is an isomorphism of abelian groups

$$\tau \circ B - \circ \eta_X : [\tilde{\Omega}X, A]_+ \rightarrow [X, MA]_+.$$

Indeed, this map is well defined thanks to Proposition 6.2, it is a bijection because of Theorem 1.19, and it is a group homomorphism because, for any $a, a' : \tilde{\Omega}X \rightarrow A$, the following diagram commutes up to based homotopy:

$$\begin{array}{ccccccc} X & \xrightarrow{\eta_X} & B\tilde{\Omega}X & \xrightarrow{B(a \cdot a')} & BA & & \\ & & \searrow^{(Ba, Ba')} & \searrow^{B(a, a')} & \downarrow B\mu_A & & \\ & & BA \times BA & \xleftarrow{(B\text{pr}_1, B\text{pr}_2)} & B(A \times A) & \xrightarrow{B\mu_A} & BA \\ & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ & & MA \times MA & \xrightarrow{\mu_{MA}} & M(A \times A) & \xrightarrow{M\mu_A} & MA \end{array}$$

Additional labels in the diagram include: $(\tau \circ Ba \circ \eta_X, \tau \circ Ba' \circ \eta_X)$ on the curved arrow from X to $MA \times MA$; $(\tau \circ Ba, \tau \circ Ba')$ on the curved arrow from $B\tilde{\Omega}X$ to $MA \times MA$; and μ_{MA} on the curved arrow from $MA \times MA$ to MA .

□

Proposition 6.5. Let X be a CW-complex pointed in a vertex, G a nice topological group, A a nice abelian group.

- (1) When $E = X$ is discrete, there are isomorphisms of abelian groups

$$[{}_X X_\star, {}_\star A_A]_+ \cong [X, A]_\star \in \mathcal{Ab},$$

where the abelian group $[X, A]_\star$ was introduced in Proposition 2.1.

- (2) When $E = G^{\text{cof}}$ is codiscrete and cofibrant, there are isomorphisms of abelian groups

$$[{}_G G^{\text{cof}}_\star, {}_\star A_A]_+ \cong [BG, MA]_\star \in \mathcal{Ab}.$$

- (3) When $E = X \times G^{\text{cof}}$ is trivial, there are isomorphisms of abelian groups

$$[{}_X X \times G^{\text{cof}}_G, {}_\star A_A]_+ \cong [X, MA]_\star \oplus [BG, MA]_\star \in \mathcal{Ab}.$$

- (4) When $E = EG$ is contractible, there are isomorphisms of abelian groups

$$[{}_G EG_G, {}_\star A_A]_+ \cong \{0\} \in \mathcal{Ab}.$$

Note that the proofs of (1), (2) and (4) are straightforward from previous results. However, there is no elementary proof of (3). We here give two arguments, but they both use results that will appear later in the paper.

Proof. (1) This is a consequence of Proposition 6.2.

(2) This is a consequence of (1) and Lemma 6.4.

(3) By Lemma 6.10, the assignments

$$(s : X \rightarrow A) \mapsto (s \circ \text{pr}_1 : X \times G \rightarrow A) \text{ and } (a : G \rightarrow A) \mapsto (a \circ \text{pr}_2 : X \times G \rightarrow A)$$

induce maps

$$[X, MA]_+ \rightarrow [X \times G, A]_+ \text{ and } [G, A]_+ \rightarrow [X \times G, A]_+.$$

They can be combined into

$$[X, MA]_+ \oplus [BG, MA]_+ \rightarrow [\star X \times G_{G, X} A]_+ \in \mathcal{Ab},$$

which is given by

$$(s : X \rightarrow A, a : G \rightarrow A) \mapsto ((s \circ \text{pr}_1) \cdot (a \circ \text{pr}_2) : X \times G \rightarrow A).$$

One can prove directly that the map is in fact a bijection. Alternatively, by using Theorem 7.1 we obtain a short exact sequence of abelian groups

$$0 \longrightarrow [X, A]_+ \xrightarrow{\pi^*} [X \times G, A]_+ \xrightarrow{\iota^*} [G, A]_+ \longrightarrow 0,$$

which splits because the mentioned above map $[G, A]_+ \rightarrow [X \times G, A]_+$ is a section.

(4) By definition, every map $EG \rightarrow A$ is of type k , and therefore plus-equivalent to the trivial map.

□

By specialising the last proposition to the case $A = M^k Z$, as a consequence of Theorem 2.8 we obtain the following.

Corollary 6.6. Let X be a CW-complex pointed in a vertex, G a nice topological group, Z an abelian countable group, and $k > 0$.

(1) When $E = X$, there are isomorphisms of abelian groups

$$H_+^k({}_X X_\star, Z)_+ \cong H^k(X; Z) \in \mathcal{Ab}.$$

(2) When $E = G$, there are isomorphisms of abelian groups

$$H_+^k({}_\star G_G, Z) \cong H_{gp}^{k+1}(G; Z) \in \mathcal{Ab}.$$

(3) When $E = X \times G$, there are isomorphisms of abelian groups

$$H_+^k({}_\star X \times G_G, Z) \cong H_{gp}^{k+1}(G; Z) \oplus H^k(X; Z) \in \mathcal{Ab}.$$

(4) When $E = EG$, there are isomorphisms of abelian groups

$$H_+^k({}_\star EG_G, Z) \cong \{\star\} \in \mathcal{Ab}.$$

□

As a consequence of Remark 5.10, if two maps of bundles are plus-equivalent, they they are also based homotopic as pointed maps of pointed spaces. Therefore we obtain the following result.

Proposition 6.7. Let A be an abelian nice topological group, G a nice topological groups or a matrix group, X a CW-complex pointed in a vertex. For every pointed G -bundle E over X there is a **comparison map** of abelian groups

$$[{}_X E_G, {}_\star A_A]_+ \rightarrow [E, A]_\star \text{ in } \mathcal{Ab}.$$

□

By specializing to the case of $A = M^k Z$, and using the fact that the cofibrant replacement does not change the homotopy type, we obtain the following corollary.

Corollary 6.8. Let Z be an abelian countable group, G a nice topological groups or a matrix group, X a CW-complex pointed in a vertex. For every pointed G -bundle E over X there is a map of abelian groups

$$H_+^k(E; Z) \rightarrow H^k(E^{\text{cof}}; Z) \text{ in } \mathcal{A}b$$

between the plus-cohomology group and the ordinary cohomology group of the cofibrant replacement. \square

Remark 6.9. The trace map is an isomorphism if $E = X$ is a discrete bundle. However it will not be in general, and not even when the bundle is codiscrete.

We now discuss the functoriality of the assignment

$$E \mapsto [E, A]_+.$$

Although the group $\mathcal{B}un(E, A)$ is functorial in E before taking the quotient modulo \simeq_* , the property for a map to be of type k is not a priori compatible with precomposition of maps. This prevents us from considering $[E, A]_+$ as a functor in $E \in \mathcal{B}un\star$. However, the assignment $[-, A]_+$ can act on at least three kinds of maps: G -equivariant morphisms of G -bundles, maps between codiscrete bundles, and maps to a discrete bundle X . This is a consequence of the following lemma. Note that the property (3) is counterintuitive.

Lemma 6.10. Let X and X' be CW-complexes pointed in vertices, G and G' nice topological groups, E a pointed G -bundle over X , E' a pointed G' -bundle over X' and A an abelian topological group. Let $\phi : E' \rightarrow E$ be a morphism in $\mathcal{B}un\star$.

- (1) If $G = G'$, for ψ in $\mathcal{B}un\star(E, A)$, if ψ is of type k , then $\psi \circ \phi$ is of type k in $\mathcal{B}un\star(E', A)$. Equivalently,

$$\phi^*(\mathcal{M}_k) \subset \mathcal{M}'_k \subset \mathcal{B}un\star(E', A).$$

- (2) If $G = G'$, for ψ in $\mathcal{B}un\star(E', A)$, if ψ is of type w , then $\psi \circ \phi$ is of type w in $\mathcal{B}un\star(E, A)$. Equivalently,

$$\phi^*(\mathcal{M}_w) \subset \mathcal{M}'_w \subset \mathcal{B}un\star(E', A).$$

- (3) If $E = X$, for ψ in $\mathcal{B}un\star(X, A)$, if ψ is of type k , then $\psi \circ \phi$ is of type w in $\mathcal{B}un\star(E', A)$. Equivalently,

$$\phi^*(\mathcal{M}_k) \subset \mathcal{M}'_w \subset \mathcal{B}un\star(E', A).$$

- (4) If $E = X$, for ψ in $\mathcal{B}un\star(X, A)$, if ψ is of type w , then $\psi \circ \phi$ is of type w in $\mathcal{B}un\star(E', A)$. Equivalently,

$$\phi^*(\mathcal{M}_w) \subset \mathcal{M}'_w \subset \mathcal{B}un\star(E', A).$$

- (5) If $E = G$ and $E' = G'$, for ψ in $\mathcal{B}un\star(G, A)$, if ψ is of type k , then $\psi \circ \phi$ is of type w in $\mathcal{B}un\star(G', A)$. Equivalently,

$$\phi^*(\mathcal{M}_k) \subset \mathcal{M}'_k \subset \mathcal{B}un\star(E', A).$$

- (6) If $E = G$ and $E' = G'$, for ψ in $\mathcal{B}un\star(G, A)$, if ψ is of type w , then $\psi \circ \phi$ is of type w in $\mathcal{B}un\star(G', A)$. Equivalently,

$$\phi^*(\mathcal{M}_w) \subset \mathcal{M}'_w \subset \mathcal{B}un\star(E', A).$$

Proof. (1) As ψ is of type k , it factors through a universal bundle \tilde{E} for G , as displayed:

$$\psi : E \xrightarrow{T} \tilde{E} \xrightarrow{\kappa} A.$$

Then $\psi \circ \phi$ factors through \tilde{E} , as displayed

$$\psi \circ \phi : E' \xrightarrow{\phi} E \xrightarrow{T} \tilde{E} \xrightarrow{\kappa} A.$$

It follows that $\psi \circ \phi$ is of type k .

- (2) As ψ is of type w, it factors through the base space X , as displayed:

$$\psi : E \xrightarrow{\pi} X \xrightarrow{s} A$$

If g is the geometric information of ϕ , then $\psi \circ \phi$ factors through the base space X' as displayed:

$$\psi \circ \phi : E' \xrightarrow{\pi'} X' \xrightarrow{g} X \xrightarrow{s} A.$$

Moreover the map s factors up to based homotopy through a classifying space \tilde{B} of G , as displayed

$$X \xrightarrow{t} \tilde{B} \xrightarrow{\bar{s}} A.$$

Then the map $s \circ g$ factors up to based homotopy through the classifying space \tilde{B} of G , as displayed

$$X' \xrightarrow{g} X \xrightarrow{t} \tilde{B} \xrightarrow{\bar{s}} A,$$

and $t \circ g$ is a classifying map for E' with respect to \tilde{E} , because of the following pointed equivalences of G -bundles over X :

$$\begin{aligned} E' &\cong g^*E \\ &\cong g^*t^*\tilde{E} \\ &\cong (t \circ g)^*\tilde{E}. \end{aligned}$$

It follows that $\psi \circ \phi$ is of type w.

- (3) As ψ is of type k, it factors through a universal bundle \tilde{E} for \star , which is a contractible space. The map ψ is therefore based homotopic to a constant map. Now, $\psi \circ \phi$ factors through the base space X' as displayed:

$$\psi \circ \phi : E' \xrightarrow{\pi'} X' \xrightarrow{g} X \xrightarrow{\psi} A.$$

Moreover the map $\psi \circ g$ is based homotopic to the trivial map because ψ is. So $\psi \circ g$ factors up to based homotopy through the classifying space BG' of G' , as displayed

$$X' \xrightarrow{t'} BG' \xrightarrow{\star} A,$$

It follows that $\psi \circ \phi$ is of type w.

- (4) As ψ is of type w, it is homotopic to a map that factors through a classifying space for \star , which is a contractible space. Then the map ψ is based homotopic to a constant map. The same argument used in (3) applies. It follows that $\psi \circ \phi$ is of type w.
- (5) As ψ is of type k, it factors through some classifying space \tilde{E} for G . As EG is a G -bundle and \tilde{E} is a universal bundle for G , there exists a G -equivariant pointed map $EG \rightarrow \tilde{E}$. Then a map κ is determined by the rest of the data and ψ also factors through EG , as displayed

$$\begin{array}{ccccc} \psi : G & \xrightarrow{\quad} & \tilde{E} & \xrightarrow{\tilde{\kappa}} & A \\ \parallel & & \uparrow & & \parallel \\ G & \xrightarrow{\quad} & EG & \xrightarrow{\kappa} & A. \end{array}$$

Then $\psi \circ \phi$ factors through $\tilde{E}G'$, as displayed

$$\psi \circ \phi : G' \xrightarrow{\quad} EG' \xrightarrow{E\psi} EG \xrightarrow{\kappa} A.$$

It follows that $\psi \circ \phi$ is of type k.

- (6) As ψ is of type w, it factors through \star , and is therefore constant. Then $\psi \circ \phi$ is constant and is in particular of type w.

□

The first consequence of Lemma 6.10 is that the assignment $E \mapsto [E, A]_+$ defines a functor on the (non-full!) subcategory $\mathcal{Bun}\star_G$ of $\mathcal{Bun}\star$ given by of pointed G -bundles and G -equivariant maps.

Proposition 6.11. Let X be a CW-complex pointed in a vertex, G a nice topological group, E a pointed G -bundle over X and A a nice abelian group. Then the assignment

$$E \mapsto [E, A]_+ := \mathcal{Bun}\star(E, A)/\simeq_+$$

defines a contravariant functor of G -bundles in abelian groups

$$[-, A]_+ : \mathcal{Bun}\star_G^{\text{op}} \rightarrow \mathcal{Ab}.$$

Proof. As a consequence of Lemma 6.10(1) and (2), plus-equivalence relation is a congruence with respect to precomposition. \square

The second consequence of Lemma 6.10 is that the assignment $E \mapsto [E, A]_+$ acts on the Puppe sequence of Remark 5.5 and produces a diagram of abelian groups, which will be proven to be an exact sequence in the next section. We will be interesting in letting $[-, A]_+$ act on the Puppe sequence of the cofibrant replacement E^{cof} of a principal bundle E .

Proposition 6.12. Let X be a nice pointed space, G a nice topological group, E a pointed G -bundle over X and A a nice abelian topological group. There is a diagram of abelian groups.

$$[BG, A]_+ \longrightarrow [X, A]_+ \longrightarrow [E^{\text{cof}}, A]_+ \longrightarrow [G^{\text{cof}}, A]_+ \longrightarrow [\tilde{\Omega}X, A]_+ .$$

\square

By specialising the two last propositions we obtain the following.

Corollary 6.13. Let X be a CW-complex pointed in a vertex, G a nice topological group or a matrix group, E a pointed G -bundle over X and Z an abelian countable group, $k > 0$. Then the assignment

$$E \mapsto H_+^k(E; Z) := [E^{\text{cof}}, M^k Z]_+$$

defines a contravariant functor of G -bundles in abelian groups

$$H_+^k(-; Z) : \mathcal{Bun}\star_G^{\text{op}} \rightarrow \mathcal{Ab}.$$

\square

Corollary 6.14. Let X be a nice pointed space, G a nice topological group or a matrix group, E a pointed G -bundle over X , $k > 0$ and Z an abelian countable group. There is a diagram of maps of abelian groups.

$$H^k(BG; Z) \longrightarrow H^k(X; Z) \longrightarrow H_+^k(E; Z) \longrightarrow H_+^k(G; Z) \longrightarrow H^k(\tilde{\Omega}X, Z) .$$

\square

7. THE CHARACTERISTIC LONG EXACT SEQUENCE

In this section we prove that the sequence of Proposition 6.12 is exact.

Theorem 7.1. Let X be a nice pointed space, G a nice topological group, E a pointed G -bundle over X and A a nice abelian topological group. There is an exact sequence of abelian groups

$$[BG, A]_+ \xrightarrow{-\circ t} [X, A]_+ \xrightarrow{-\circ \pi} [E, A]_+ \xrightarrow{-\circ \iota} [G, A]_+ \xrightarrow{-\circ b} [\tilde{\Omega}X, A]_+ .$$

We need a preliminary lemma.

Lemma 7.2. Let X be a CW-complex pointed in a vertex, G a nice topological group, E a pointed G -bundle over X and A a nice abelian topological group. Every a -equivariant pointed morphism $\psi : E \rightarrow A$ with $a \equiv_* e_A : G \rightarrow A$ admits an ik-decomposition

$$\psi = \psi_i \cdot \psi_k \in \mathcal{Bun}\star(E, A).$$

In the proof, if $\psi : E \rightarrow A$ is a map of bundles, we denote by $\psi^{-1} : E \rightarrow A$ the map obtained by inverting pointwise in A the map ψ .

Proof of Lemma 7.2. We first fix a G -equivariant pointed map $T : E \rightarrow EG$, that induces a classifying map $t : X \rightarrow BG$. In particular there is a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ T \downarrow & & \downarrow t \\ EG & \longrightarrow & BG. \end{array}$$

As $a \equiv_* e_A$, by Lemma 6.3 it extends to a pointed a -equivariant map $\kappa : EG \rightarrow A$. Recalling the proof of Lemma 6.3, we see that κ factors through a_*EG , as displayed

$$\kappa : EG \xrightarrow{\eta_a} a_*EG \xrightarrow{\bar{\kappa}} A,$$

and the map $\bar{\kappa}$ induces a pointed equivalence of A -bundles over BG

$$(\pi', \bar{\kappa}) : a_*EG \rightarrow BG \times A.$$

On the other hand, by Proposition 1.12 the ψ factors through a_*E , as displayed

$$\psi : E \xrightarrow{\eta_a} a_*E \xrightarrow{\bar{\psi}} A,$$

and the map $\bar{\psi}$ induces a pointed equivalence of A -bundles over X

$$(\pi, \bar{\psi}) : a_*E \rightarrow X \times A.$$

These maps fit in a commutative diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\hspace{10em}} & A \\ \parallel & \eta_a \searrow & & \bar{\psi} \searrow & \parallel \\ E & \xrightarrow{\hspace{10em}} & a_*E & \xrightarrow{\hspace{10em}} & A \\ \parallel & \eta_a \searrow & \parallel & & \parallel \\ E & \xrightarrow{\hspace{10em}} & a_*E & \xrightarrow[\cong]{(\bar{\pi}, \bar{\kappa} \circ a_*T)} & X \times A & \dashrightarrow & A \\ T \downarrow & & a_*T \downarrow & & \downarrow t \times A & & \parallel \\ EG & \xrightarrow{\hspace{10em}} & a_*EG & \xrightarrow[\cong]{(\pi', \bar{\kappa})} & BG \times A & & A. \end{array}$$

The map $X \times A \rightarrow A$ is obtained by composing the inverse of $(\bar{\pi}, \bar{\kappa} \circ a_*T)$ and $\bar{\psi}$. As it is A -equivariant, it must be of the form

$$s \cdot A : X \times A \xrightarrow{s \times A} A \times A \xrightarrow{\cdot} A,$$

where $s : X \rightarrow A$ is a pointed map.

There is no reason to believe that $s \cdot A$ should factor through $BG \times A$. Using the fact that both ψ and $(s \circ \pi)^{-1}$ factor through a_*E and $X \times A$, we write instead a similar diagram for

$$\psi_k := \psi \cdot (s \circ \pi)^{-1} : E \rightarrow A.$$

The key fact is that the remaining map $X \times A \rightarrow A$ is now just a projection on the second factor. In particular it factors through $BG \times A$, as displayed

$$\begin{array}{ccccccc}
E & \xrightarrow{\psi_k = \psi \cdot (s \circ \pi)^{-1}} & & & A \\
\parallel & \eta_a \searrow & a_* E & \xrightarrow{\bar{\psi} \cdot (s \circ \pi)^{-1}} & A \\
\parallel & & \parallel & & \parallel \\
E & \xrightarrow{\eta_a} & a_* E & \xrightarrow[\cong]{(\pi, \bar{\kappa} \circ a_*)} & X \times A & \xrightarrow[\text{pr}_2]{(s \cdot A) \cdot (s \circ \text{pr}_1)^{-1}} & A \\
T \downarrow & & a_* T \downarrow & & \downarrow t \times A & & \parallel \\
EG & \xrightarrow{\eta_a} & a_* EG & \xrightarrow[\cong]{(\pi', \bar{\kappa})} & BG \times A & \xrightarrow{\text{pr}_2} & A.
\end{array}$$

As a consequence ψ_k factors through the universal bundle EG for G , and is of type k.

We then set

$$\psi_i := s \circ \pi : E \rightarrow A,$$

which is of type i, and the desired factorization of ψ follows

$$\psi = (s \circ \pi) \cdot (\psi \cdot (s \circ \pi)^{-1}) = \psi_i \cdot \psi_k.$$

□

Proof of Theorem 7.1. We prove the exactness in $[X, A]_+$: In order to show that $\text{im}(t^*) \subset \ker(\pi^*)$, take $[s : BG \rightarrow A]_+$. As $s \circ t \circ \pi$ obviously factors through BG

$$s \circ t \circ \pi : E \xrightarrow{\pi} X \xrightarrow{t} BG \xrightarrow{s} A,$$

it is of type ik and

$$\pi^*(t^*([s]_+)) = [s \circ t \circ \pi]_+ = [e_A]_+,$$

namely $t^*([s]_+) \in \ker(\pi^*)$.

In order to show that $\text{im}(t^*) \supset \ker(\pi^*)$, take $[s : X \rightarrow A]_+ \in \ker(\pi^*)$. Then

$$[e_A]_+ = \pi^*([s]_+) = [s \circ \pi]_+.$$

Then there exists $\psi_w : E \rightarrow A$ of type w and $\psi_k : E \rightarrow A$ of type k such that $s \circ \pi = \psi_w \cdot \psi_k$. As ψ_k is of type k, it factors through a universal bundle \tilde{E} for G . Moreover $s \circ \pi$ and ψ_w factor through X , therefore their algebraic information is trivial, and it follows that also the algebraic information $a(\psi_k)$ of ψ_k is trivial,

$$a(\psi_k) = a((s \circ \pi) \cdot \psi_w^{-1}) = a(s \circ \pi) \cdot a(\psi_w)^{-1} = e_A \cdot e_A = e_A.$$

Thus ψ_k factors through the quotient $\tilde{B} := \tilde{E}/G$, as displayed

$$\begin{array}{ccccccc}
E & \xrightarrow{\psi_k} & & & A \\
\parallel & T \searrow & \tilde{E} & \xrightarrow{\quad} & A \\
\parallel & & \parallel & & \parallel \\
E & \xrightarrow{T} & \tilde{E} & \xrightarrow{\quad} & \tilde{B} & \xrightarrow{s_k} & A \\
\parallel & & \parallel & & \parallel & & \parallel \\
E & \xrightarrow{\pi} & X & \xrightarrow{t} & \tilde{B} & \xrightarrow{s_k} & A \\
\parallel & T \searrow & \parallel & & \parallel & & \parallel \\
E & \xrightarrow{T} & X & \xrightarrow{g_k} & \tilde{B} & \xrightarrow{s_k} & A.
\end{array}$$

Here, the maps s_k and g_k are determined by the rest of the data. On the other hand, as ψ_w is of type w, it factors through the base space X and it factors up to based homotopy through any

classifying space for G , and in particular \tilde{B} , as displayed

$$\begin{array}{ccccc} E & \xrightarrow{\psi_w} & & & A \\ \parallel & & & & \parallel \\ E & \xrightarrow{\pi} & X & \xrightarrow{g_w} & A \\ \parallel & & \parallel & \simeq_* & \parallel \\ E & \xrightarrow{\pi} & X & \xrightarrow{t} & \tilde{B} \xrightarrow{s_w} A. \end{array}$$

Here, the maps g_w and s_w exist because ψ_w is of type w . Therefore

$$s \circ \pi = \psi_k \cdot \psi_w = (g_k \circ \pi) \cdot (g_w \circ \pi) = (g_k \cdot g_w) \circ \pi.$$

and, since π is surjective,

$$s = g_k \cdot g_w = (s_k \circ t) \cdot g_w \simeq_* (s_k \circ t) \cdot (s_w \circ t) = (s_k \cdot s_w) \circ t.$$

Finally

$$[s]_+ = [(s_k \cdot s_w) \circ t]_+ = t^*([s_k \cdot s_w]_+) \in \text{im}(t^*).$$

We prove the exactness in $[E, A]_+$. In order to show that $\text{im}(\pi^*) \subset \ker(\iota^*)$, take $[s : X \rightarrow A]_+ \in [X, A]_+$ and remark that $\pi \circ \iota$ is constant. Therefore

$$\iota^*(\pi^*([s]_+)) = [s \circ \pi \circ \iota]_+ = [s \circ e_A]_+ = [e_A]_+,$$

namely $\pi^*[s] \in \ker(\iota^*)$.

In order to show that $\ker(\iota^*) \subset \text{im}(\pi^*)$, take $[\psi : E \rightarrow A]_+ \in \ker(\iota^*)$. Then the algebraic information of ψ is plus-equivalent to the trivial homomorphism,

$$[e_A]_+ = \iota^*([\psi]_+) = [\iota \circ \psi]_+ = [a(\psi)]_+.$$

By Lemma 7.2, there exists $\psi_i : E \rightarrow A$ of type i , and $\psi_k : E \rightarrow A$ of type k such that $\psi = \psi_i \cdot \psi_k$. As ψ_i is of type i , it factors through the base space X

$$\psi_i : E \xrightarrow{\pi} X \xrightarrow{s_i} A.$$

In particular $\psi_k \simeq_+ e_A$, and

$$[\psi]_+ = [\psi_i \cdot \psi_k]_+ = [\psi_i]_+ \cdot [\psi_k]_+ = [\psi_i]_+ \cdot e_A = [\psi_i]_+ = [s_i \circ \pi]_+ = \pi^*([s_i]_+) \in \text{im}(\pi^*).$$

We prove the exactness in $[G, A]_+$. In order to show that $\text{im}(\iota^*) \subset \ker(b^*)$, take $[\psi : E \rightarrow A]_+$. By Theorem 1.19, there exists a classifying map $b : \tilde{\Omega}X \rightarrow G$ and an a -equivariant map $B : \tilde{P}X \rightarrow E$. In particular there is a commutative diagram

$$\begin{array}{ccc} \tilde{\Omega}X & \xrightarrow{\tilde{\iota}} & \tilde{P}X \\ b \downarrow & & \downarrow B \\ G & \xrightarrow{\iota} & E. \end{array}$$

It follows that $\psi \circ \iota \circ b$ factors through the universal bundle $\tilde{P}X$ for $\tilde{\Omega}X$,

$$\begin{array}{ccccccc} \psi \circ \iota \circ b : \tilde{\Omega}X & \xrightarrow{b} & G & \xrightarrow{\iota} & E & \xrightarrow{\psi} & A \\ \parallel & & & & \parallel & \psi & \parallel \\ \tilde{\Omega}X & \xrightarrow{\tilde{\iota}} & \tilde{P}X & \xrightarrow{B} & E & \xrightarrow{\psi} & A. \end{array}$$

Then it is of type k and

$$b^*(\iota^*([\psi]_+)) = [\psi \circ \iota \circ b]_+ = [e_A]_+.$$

Therefore $\iota^*([\psi]_+) \in \ker(b^*)$.

In order to show that $\ker(b^*) \subset \text{im}(\iota^*)$, take $[a : G \rightarrow A]_+ \in \ker(b^*)$. Then

$$[e_A]_+ = b^*([a]_+) = [a \circ b]_+,$$

and, by Proposition 6.2(2), $a \circ b \equiv_* e_A$. There are then pointed equivalences of A -bundles over X

$$\begin{aligned} a_* E &\cong a_* b_* \tilde{P}X \\ &\cong (a \circ b)_* \tilde{P}X \\ &\cong e_{A*} \tilde{P}X \\ &= X \times A, \end{aligned}$$

that we use to construct the map

$$\psi : E \rightarrow a_* E \cong X \times A \rightarrow A,$$

whose algebraic information is $a : G \rightarrow A$. This means that

$$\text{im}(\iota^*) \ni \iota^*([\psi]_+) = [\psi \circ \iota]_+ = [a(\psi)]_+ = [a]_+.$$

□

Corollary 7.3. Let X be a nice pointed space, G a nice topological group or a matrix group, E a pointed G -bundle over X , Z an abelian countable group and $k > 0$. There is a long exact sequence of abelian groups

$$\dots \longrightarrow H^k(X; Z) \longrightarrow H_+^k(E; Z) \longrightarrow H_{\mathcal{G}^p}^{k+1}(G; Z) \xrightarrow{\chi_{k+1}} H^{k+1}(X; Z) \longrightarrow \dots$$

where the connecting map is the characteristic map of E .

Note that this long exact sequence is obtained by dualizing the classical argument used to obtain the homotopy long exact sequence of Theorem 1.9.

Proof. We first notice that the map $\tilde{\Omega}t : \tilde{\Omega}X \rightarrow \tilde{\Omega}BG = G^{\text{cof}}$, constructed in Remark 1.20, is the classifying map of E^{cof} , as there are pointed equivalences of pointed G^{cof} -bundles over X

$$E^{\text{cof}} := t^* \tilde{P}BG \cong (\tilde{\Omega}t)_* \tilde{P}X.$$

By applying Theorem 7.1 of E^{cof} with $k > 0$ and $A := M^k Z$ we obtain a sequence of abelian groups with five terms:

$$[BG, M^k Z]_+ \xrightarrow{-\circ t} [X, M^k Z]_+ \longrightarrow [E^{\text{cof}}, M^k Z]_+ \longrightarrow [G^{\text{cof}}, M^k Z]_+ \xrightarrow{-\circ \tilde{\Omega}t} [\tilde{\Omega}X, M^k Z]_+.$$

By Lemma 6.4, there is a commutative diagram

$$\begin{array}{ccc} [\tilde{\Omega}BG, M^k Z]_+ & \xrightarrow{-\circ \tilde{\Omega}t} & [\tilde{\Omega}X, M^k Z]_+ \\ \cong \downarrow & & \cong \downarrow \\ [BG, M^{k+1} Z]_+ & \xrightarrow{-\circ t} & [X, M^{k+1} Z]_+ \\ \cong \downarrow & & \cong \downarrow \\ H^{k+1}(BG; Z) & \xrightarrow{\chi_{k+1}} & H^{k+1}(X; Z). \end{array}$$

Therefore the connecting map to the right in degree k ,

$$-\circ \tilde{\Omega}t : [G^{\text{cof}}, M^k Z]_+ \rightarrow [\tilde{\Omega}X, M^k Z]_+,$$

can be identified with the connecting map map to the left in degree $k+1$,

$$-\circ t : [BG, M^{k+1} Z]_+ \rightarrow [X, M^{k+1} Z]_+,$$

which can then be identified with the characteristic map of E in degree $k+1$,

$$\chi_{k+1} : H^{k+1}(BG; Z) \rightarrow H^{k+1}(X; Z).$$

$$\begin{array}{ccccccccc} [BG, M^k Z]_+ & \longrightarrow & [X, M^k Z]_+ & \longrightarrow & [E^{\text{cof}}, M^k Z]_+ & \longrightarrow & [G^{\text{cof}}, M^k Z]_+ & \longrightarrow & [\tilde{\Omega}X, M^k Z]_+ \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ H_{\mathcal{G}^p}^k(G; Z) & \longrightarrow & H^k(X; Z) & \longrightarrow & H_+^k(E; Z) & \longrightarrow & H_{\mathcal{G}^p}^{k+1}(G; Z) & \longrightarrow & H^{k+1}(X; Z) \end{array}$$

Starting with $k = 0$ we can extend the sequence to the right infinitely many times, and obtain the desired long exact sequence. \square

Combining Corollaries 6.5 and 7.3, we obtain two special cases of the sequence above.

Corollary 7.4. Let X be a nice pointed space and G a topological group or a matrix group, and Z be an abelian countable group.

» If $E = X \times G$ is the trivial bundle, then the long exact sequence of Corollary 7.3 becomes

$$\dots \rightarrow H^k(X; Z) \rightarrow H^k(X; G) \oplus H^{k+1}(BG; Z) \rightarrow H_{\mathcal{G}p}^{k+1}(G; Z) \xrightarrow{*} H^{k+1}(X; Z) \rightarrow \dots$$

» If $E = EG$ is a contractible bundle, then the long exact sequence of Corollary 7.3 becomes

$$\dots \rightarrow H^k(X; Z) \rightarrow \{\star\} \rightarrow H_{\mathcal{G}p}^{k+1}(G; Z) \xrightarrow{\cong} H^{k+1}(X; Z) \rightarrow \dots$$

\square

Given a bundle, by letting the ordinary cohomology groups act on the cofiber sequence

$$G \xrightarrow{\iota} E \rightarrow C(\iota) \rightarrow \Sigma G.$$

of the Hurewicz cofibration $G \hookrightarrow E$, there is a long cohomology exact sequence [Pic92]

$$\dots \rightarrow H^k(X; Z) \xrightarrow{H^k(\pi; Z)} H^k(E; Z) \xrightarrow{H^k(\iota; Z)} H^k(G; Z) \rightarrow H^{k+1}(C(\iota); Z) \rightarrow \dots$$

Note that X is different from the homotopy quotient $C(\iota)$, obtained by collapsing only the distinguished fiber to a point. We conclude by proving that this sequence can be compared to the sequence of Corollary 7.3.

Proposition 7.5. Let X be a nice pointed space, G a topological group or a matrix group, E a pointed G -bundle over X , Z an abelian countable group and $k > 0$. The comparison map of Corollary 6.8 induces a morphism of long exact sequences of abelian groups

$$\begin{array}{ccccccc} \dots & \rightarrow & H^k(X; Z) & \rightarrow & H_+^k(E; Z) & \rightarrow & H_{\mathcal{G}p}^{k+1}(G; Z) \xrightarrow{\chi_{k+1}} H^{k+1}(X; Z) \rightarrow \dots \\ & & \downarrow H^k(C(\iota) \rightarrow X; Z) & & \downarrow & & \downarrow H^{k+1}(C(\iota) \rightarrow X; Z) \\ \dots & \rightarrow & H^k(C(\iota); Z) & \xrightarrow{H^k(\pi; Z)} & H^k(E^{\text{cof}}; Z) & \xrightarrow{H^k(\iota; Z)} & H^k(G^{\text{cof}}; Z) \rightarrow H^{k+1}(C(\iota); Z) \rightarrow \dots \end{array}$$

Proof. We prove that the square involving X and G commutes. There is a homotopy commutative diagram of pointed spaces (of which a detailed construction can be found in [Rovb, Section 5.1]):

$$(*) \quad \begin{array}{ccc} X & \xrightarrow{t} & BG^{\text{cof}} \\ \uparrow & & \uparrow \\ C(\iota) & \longrightarrow & \Sigma G^{\text{cof}}. \end{array}$$

We note that there are isomorphisms of abelian groups

$$[\Sigma G^{\text{cof}}, M^{k+1}Z]_* \cong [G^{\text{cof}}, \Omega M^{k+1}Z]_* \cong H^k(G^{\text{cof}}; Z).$$

Therefore, by applying $[-, M^{k+1}Z]_*$ to the last square of $(*)$ we are done.

We prove that the square involving G and E commutes. The comparison map of Corollary 6.8 is natural on morphisms of G -bundles for a fixed G . In particular, the naturality on the inclusion $G^{\text{cof}} \hookrightarrow E^{\text{cof}}$ yields the desired square.

We prove that the square involving E and X commutes. The comparison map of Corollary 6.8 is natural on the projection $E^{\text{cof}} \rightarrow X$. Thus we have a commutative diagram

$$\begin{array}{ccc}
 H_+^k(X; Z) & \xrightarrow{H_+^k(\pi; Z)} & H_+^k(E; Z) \\
 \cong \downarrow & & \downarrow \\
 H^k(X; Z) & \xrightarrow{H^k(\pi; Z)} & H^k(E^{\text{cof}}; Z), \\
 \searrow H^k(C(\iota) \rightarrow X; Z) & & \nearrow H^k(E^{\text{cof}} \rightarrow C(\iota); Z) \\
 & H^k(C(\iota); Z) &
 \end{array}$$

which concludes the proof. \square

REFERENCES

- [BH98] C. Berger and J. Huebschmann, *Comparison of the geometric bar and W -constructions*, J. Pure Appl. Algebra **131** (1998), no. 2, 109–123. MR 1637515 (99e:18010)
- [Dol63] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) **78** (1963), 223–255. MR 0155330
- [DS95] W.G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR 1361887
- [Far96] E.D. Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR 1392221
- [FP90] R. Fritsch and R.A. Piccinini, *Cellular structures in topology*, Cambridge Studies in Advanced Mathematics, vol. 19, Cambridge University Press, Cambridge, 1990. MR 1074175
- [Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)
- [Hir] P.S. Hirshhorn, *Overcategories and undercategories of model categories*, online article.
- [May72] J.P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972, Lectures Notes in Mathematics, Vol. 271. MR 0420610
- [May99] ———, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR 1702278
- [Mil56a] J. Milnor, *Construction of universal bundles. I*, Ann. of Math. (2) **63** (1956), 272–284. MR 0077122 (17,994b)
- [Mil56b] ———, *Construction of universal bundles. II*, Ann. of Math. (2) **63** (1956), 430–436. MR 0077932 (17,1120a)
- [Mil67] R.J. Milgram, *The bar construction and abelian H -spaces*, Illinois J. Math. **11** (1967), 242–250. MR 0208595 (34 #8404)
- [Mit11] S. Mitchell, *Notes on principal bundles and classifying spaces*, 2011, online notes.
- [ML70] S. Mac Lane, *Milgram’s classifying space as a tensor product of functors*, The Steenrod Algebra and its Applications, Lecture Notes in Mathematics **168** (1970), 135–152.
- [ML98] ———, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872 (2001j:18001)
- [MS74] J.W. Milnor and J.D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76. MR 0440554
- [Mun75] J.R. Munkres, *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975. MR 0464128
- [NSS15] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞ -bundles: general theory*, J. Homotopy Relat. Struct. **10** (2015), no. 4, 749–801. MR 3423073
- [Pic92] R.A. Piccinini, *Lectures on homotopy theory*, North-Holland Mathematics Studies, vol. 171, North-Holland Publishing Co., Amsterdam, 1992. MR 1150242 (93e:55001)
- [Rie14] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014. MR 3221774
- [Rova] M. Rovelli, *A looping-delooping adjunction for topological spaces*, to appear in Homology, Homotopy and Applications.
- [Rovb] ———, *Towards new invariants for principal bundles*, Ph.D. thesis, in preparation.
- [Seg74] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 0353298
- [Spa81] E.H. Spanier, *Algebraic topology*, Springer-Verlag, New York-Berlin, 1981, Corrected reprint. MR 666554 (83i:55001)
- [SSS09] H. Sati, U. Schreiber, and J. Stasheff, *Fivebrane structures*, Rev. Math. Phys. **21** (2009), no. 10, 1197–1240. MR 2588823
- [Sta78] J.D. Stasheff, *Continuous cohomology of groups and classifying spaces*, Bull. Amer. Math. Soc. **84** (1978), no. 4, 513–530. MR 0494071
- [Ste99] N. Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, Reprint of the 1957 edition, Princeton Paperbacks. MR 1688579 (2000a:55001)

- [Str] N. Strickland, *The category of CGWH spaces*, online article.
- [Whi49] J.H.C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245. MR 0030759 (11,48b)
- [Whi52] G.W. Whitehead, *Fiber spaces and the Eilenberg homology groups*, Proc. Nat. Acad. Sci. U. S. A. **38** (1952), 426–430. MR 0048029

(M. Rovelli) DEPARTMENT OF MATHEMATICS, EPF LAUSANNE, SWITZERLAND
E-mail address: `martina.rovelli@epfl.ch`